AN INTRODUCTION
TO LINEAR DIFFERENCE EQUATIONS

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BY

PAUL M. BATCHELDER, Ph. D.
ADJUNCT PROFESSOR OF PURE MATHEMATICS
THE UNIVERSITY OF TEXAS

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# PREFACE.

Important progress has been made during recent years in the theory of linear difference equations. Previously the development of this subject had lagged far behind that of the related field of linear differential equations, as a result of certain intrinsic difficulties which called for the introduction of new ideas and methods. The first of these new ideas was that of asymptotic representation, which was developed by Poincaré and applied by him in 1885 to the study of difference equations.* About 1910 effective methods of attack were devised almost simultaneously by Nörlund in Denmark, Carmichael and Birkhoff in this country, and Galbrun in France.† These four mathematicians all succeeded in proving in different ways the existence of analytic solutions of linear homogeneous difference equations and studying their properties. Since then considerable further work has been done along the same lines both by these and by other investigators.

Up to the present time no attempt has been made to provide the student with a convenient introduction to this new field, in which so many problems still await solution. The only book which deals with the theory from the modern standpoint is the recent one of Nörlund;‡ while this is of great value to the advanced student in furnishing a general view of the literature of the subject, its failure to give a

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* *Amer. Journ. Math.*, 7 (1885), pp. 203–258.

† Nörlund: *Bidrag til de lineaere Differensligningers Theori* (Copenhagen, 1910). Carmichael: *Trans. Amer. Math. Soc.*, 12 (1911), pp. 99–134. Birkhoff: *ibid.*, pp. 243–284. Galbrun: *Acta Math.*, 36 (1912), pp. 1–68.

‡ *Vorlesungen über Differenzenrechnung* (Berlin, 1924).

systematic presentation of the elements of the theory renders it unsuitable for the beginner. The aim of the present book is in part to fill this need by affording as simple and direct an approach as possible to the fundamental facts and ideas of the theory, and in part to extend the boundaries of the subject by studying certain important exceptional cases which have hitherto defied analysis.

No knowledge is presupposed of the student beyond the elements of the theory of functions of a complex variable. In the first chapter various topics relating to difference equations are dealt with, and the reader becomes acquainted with some of the fundamental concepts and methods which he will need later. The second chapter is devoted to difference equations of the first order, including of course the well known gamma function, all of whose principal properties are derived in a simple and direct fashion. To increase the value of these two chapters to the student a set of exercises is added at the end of each.

In the third and fourth chapters a detailed study is made of the hypergeometric difference equation, i. e., the linear homogeneous equation of the second order with linear coefficients. The theory of this equation presents most of the main points of interest of the general theory of linear homogeneous equations of the $n$th order with rational coefficients, yet without the excessive complication and abstractness of the latter, because concrete and explicit formulas can be obtained throughout. For this reason it affords an admirable introduction to the general theory. The form of treatment employed aims to familiarize the student both with the methods of Birkhoff, who bases his work on the divergent power series which formally satisfy the difference equation, and with those of Nörlund, who makes extensive use of the Laplace transformation and of factorial series and other expansions. For similar reasons the hypergeometric equation is sometimes used in the form of a single equation of the second order, and sometimes as a system of two equations of the first order, each form having certain advantages which the other lacks.

The third chapter deals with what is called the "general case" of the hypergeometric equation, in which the two roots of the characteristic equation are assumed to be finite, distinct from each other, and different from zero. It is this case which has almost exclusively been treated hitherto. In the fourth chapter the theory is extended as far as possible to the "irregular cases", in which the above conditions are not satisfied. It is in this chapter that most of the new results in the present book are contained. It has been found possible to work out the theory of the cases where one root of the characteristic equation is zero, or one infinite, or both, almost as completely as that of the general case. In the remaining cases, in which the roots are equal to each other, considerable progress has been made, but further investigation is still required before the theory can be considered complete.

The author desires to express his gratitude to Prof. C. R. Adams of Brown University for reading the manuscript of this book and making numerous helpful suggestions, and to the National Research Council for its aid in publishing the work. Above all is he deeply indebted to his friend and former teacher, Prof. G. D. Birkhoff of Harvard University, under whose guidance he first studied difference equations. The general spirit of the method of treatment adopted in the first two chapters, as well as many of the details, are due to him, and his generous aid and counsel have been a constant inspiration to the author throughout.

P. M. BATCHELDER.

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### THE HYPERGEOMETRIC EQUATION: IRREGULAR CASES.

## Chapter I.

# Fundamental ideas.

## § 1. Difference equations.

A *difference equation* is a functional equation of the form

$$\tag{1} \begin{aligned} \Phi(x, f(x), f(x+1), \cdots f(x+n), \\ g(x), \cdots, g(x+m), \cdots, k(x+l)) = 0, \end{aligned}$$

where $\Phi$ is a given function of the independent variable $x$ and the values of the unknown functions $f(x)$, $g(x)$, $\cdots$, $k(x)$ at the set of points $x$, $x+1$, $x+2$, $\cdots$. The more general case in which the interval between the successive points is any real or complex number $\omega$, instead of 1, can be reduced at once to that above by the substitution $x = \omega x'$, $f(\omega x') = \varphi(x')$. The variable $x$ we shall take to be complex: $x = u + iv$ ($u$, $v$ real, $i = \sqrt{-1}$).

A difference equation in which the values of one or more of the unknown functions at $x$ and $x+n$ both appear, but not their values at $x+n+1$, $x+n+2$, $\cdots$, is said to be *of the nth order*.

If $\Phi$ is linear in all its arguments except $x$, the equation is called *linear*. We shall consider only linear equations in this book. The general linear difference equation of the $n$th order in one unknown function $y(x)$ may be written

$$\tag{2} \begin{aligned} a_n(x)\,y(x+n) + a_{n-1}(x)\,y(x+n-1) + \cdots + a_0(x)\,y(x) \\ = b(x), \end{aligned}$$

where the coefficients $a_n(x)$, $a_{n-1}(x)$, $\cdots$, $a_0(x)$, $b(x)$ are known functions. If $b(x) = 0$, the equation is *homogeneous*, otherwise it is *non-homogeneous*.

The function

$$\Delta y(x) = y(x+1) - y(x)$$

is called the *difference* of $y(x)$. The difference of this, namely

$$\Delta^2 y(x) = \Delta(\Delta y(x)) = y(x+2) - 2y(x+1) + y(x),$$

is called the *second difference* of $y(x)$, etc. The $n$th difference is

$$\Delta^n y(x) = y(x+n) - ny(x+n-1) + \frac{n(n-1)}{2} y(x+n-2) - \cdots + (-1)^n y(x).$$

These equations may be solved for $y(x+1), y(x+2), \cdots, y(x+n)$, giving

$$y(x+1) = y(x) + \Delta y(x),$$
$$y(x+2) = y(x) + 2\Delta y(x) + \Delta^2 y(x),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y(x+n) = y(x) + n\Delta y(x) + \frac{n(n-1)}{2}\Delta^2 y(x) + \cdots + \Delta^n y(x).$$

If these are substituted in eq. (2) the latter takes the form

$$A_n(x)\Delta^n y(x) + A_{n-1}(x)\Delta^{n-1} y(x) + \cdots + A_0(x) y(x) = b(x).$$

It was this notation which led to the name "difference equation".

Instead of a single linear equation of the $n$th order, we may have a system of $n$ simultaneous linear equations of the first order

$$(3) \qquad y_i(x+1) = \sum_{j=1}^{n} a_{ij}(x) y_j(x) + b_i(x) \quad (i = 1, 2, \cdots, n).$$

involving $n$ unknown functions $y_i(x)$. Such a system is essentially equivalent to eq. (2). For many purposes it is more convenient than a single equation; it has greater symmetry, and with the aid of the matrix and determinant notations it brings out clearly the analogy between $n$th order and first order equations. On the other hand, the single equation is superior in that a single unknown function is

isolated and its explicit dependence on the known coefficients is stated. In the present book we shall not confine ourselves exclusively to either form. For $n = 1$ the two forms coincide, and we have, as the simplest example of both, the equation

$$y(x+1)-r_1(x)y(x) = r_2(x), \tag{4}$$

which is studied in Chapter II.

For the most part we shall consider only difference equations whose coefficients are rational functions. In seeking solutions of such equations we shall limit our attention to functions which are analytic in the finite part of the complex plane except for poles.

## § 2. Periodic functions. Summation.

Consider the special case $r_1(x) = 1$, $r_2(x) = 0$ of eq. (4), namely

$$y(x+1)-y(x) = 0. \tag{5}$$

This is obviously satisfied by every function whose value at $x+1$ is the same as its value at $x$, i. e., by every periodic function of period 1. Periodic functions of unit period are of great importance in the theory of difference equations, and play there a rôle analogous to that of simple constants in the theory of differential equations. Eq. (5) may be written $\Delta y(x) = 0$, which shows that these periodic functions are functions whose difference is zero, just as constants are functions whose derivative is zero.

A periodic function of period 1 is determined over the whole plane by its values in a vertical strip of unit width, say the strip between the axis of imaginaries and a line parallel to it one unit to the right (including one boundary but not the other). Such a strip constitutes a *fundamental region* for the periodic function. The limitation imposed at the end of § 1 will make it unnecessary for us to consider any periodic functions except such as are analytic, apart from poles, in the finite part of a periodic strip. The simplest analytic function, other than a constant, which has the period

1 is $e^{2\pi i x}$. Most of the periodic functions whose explicit forms we shall obtain will prove to be rational functions of $e^{2\pi i x}$. Throughout this book, when periodic functions are mentioned it is to be understood that the period is 1, unless a different period is explicitly stated.

It is evident that if the values of a periodic function $p(x)$ are considered along a line parallel to the axis of reals, the same values are repeated over and over as $x$ moves to the right or left. Hence if $p(x)$ approaches a constant limit as $x \to \infty$ along every such line, we can conclude that it is identically equal to this constant.

The following theorem from the theory of functions of a complex variable, which we quote for later reference, is useful in finding the explicit forms of certain periodic functions which appear in the course of our work.*

THEOREM. *If a periodic function of period* 1 *has no singularities except poles in a fundamental period strip, and approaches a definite value, finite or infinite, at each end of the strip, it is a rational function of* $e^{2\pi i x}$.

Consider now the equation

$$y(x+1) - y(x) = \varphi(x); \tag{6}$$

this states that the difference $\Delta y(x)$ of the unknown function is equal to the given function $\varphi(x)$. Hence the problem of solving it is equivalent to that of finding a function whose difference is known. If we denote by $\mathop{\mathsf{S}}$ the operation which is the inverse of that indicated by $\Delta$, we may write

$$y(x) = \mathop{\mathsf{S}} \varphi(x) \tag{7}$$

as a solution of eq. (6). This operation is known as *summation*; it is analogous to integration, and for that reason is sometimes called "finite integration". Eq. (7) is a solution

* Cf. Burkhardt-Rasor: *Theory of Functions of a Complex Variable*, p. 275; or Osgood: *Lehrbuch der Funktionentheorie*, I (2nd ed.), p. 466.

of eq. (6) in the same sense that $\int \varphi(x)\, dx$ is a solution of the differential equation $dy/dx = \varphi(x)$. The evaluation of the sum, like that of the integral, is often a problem of great difficulty; in fact, the sum may not exist, just as the integral may not exist.

If $y_1(x)$ and $y_2(x)$ are any two solutions of eq. (6), we see by substituting them in (6) and subtracting that

$$y_1(x+1) - y_2(x+1) - [y_1(x) - y_2(x)] = 0,$$

i. e., their difference is a periodic function. The most general solution of (6) is therefore obtained by adding an arbitrary periodic function to any particular solution. This arbitrary periodic function corresponds to the constant of integration which is added to an indefinite integral.

If $\varphi(x)$ is a polynomial, its sum is a polynomial of the next higher degree, which can be obtained in any particular case by the method of undetermined coefficients. Following are a few examples of sums; they may easily be verified by forming the difference of the right-hand side.

$$\mathop{\large\mathrm{S}} 1 = x; \qquad \mathop{\large\mathrm{S}} x = \frac{x(x-1)}{2};$$

$$\mathop{\large\mathrm{S}} \binom{x}{n} = \binom{x}{n+1}^{*}; \qquad \mathop{\large\mathrm{S}} a^x = \frac{a^x}{a-1};$$

$$\mathop{\large\mathrm{S}} e^{2n\pi i x} = x e^{2n\pi i x}; \qquad \mathop{\large\mathrm{S}} \sin x = -\frac{\cos(x-\frac{1}{2})}{2\sin\frac{1}{2}}.$$

One method of evaluating the sum $\mathop{\large\mathrm{S}} \varphi(x)$ is as follows. By replacing $x$ successively by $x-1$, $x-2$, etc. in eq. (6), we see that

$$\begin{aligned} y(x) &= \varphi(x-1) + y(x-1) \\ &= \varphi(x-1) + \varphi(x-2) + y(x-2) \\ &\quad \cdots\cdots\cdots\cdots\cdots \end{aligned}$$

(8) $$= \varphi(x-1) + \varphi(x-2) + \cdots + \varphi(x-n) + y(x-n);$$

* $\binom{x}{n}$ denotes the expression $\dfrac{x(x-1)\cdots(x-n+1)}{n!}$.

similarly,

$$
\begin{aligned}
y(x) &= -\varphi(x) + y(x+1) \\
&= -\varphi(x) - \varphi(x+1) + y(x+2) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&= -\varphi(x) - \varphi(x+1) - \cdots - \varphi(x+n) + y(x+n+1).
\end{aligned} \tag{9}
$$

If now we let $n \to \infty$, we obtain the two series

$$
y_l(x) = \sum_{n=1}^{\infty} \varphi(x-n), \qquad y_r(x) = -\sum_{n=0}^{\infty} \varphi(x+n), \tag{10}
$$

which formally satisfy eq. (6). They may be termed the *symbolic solutions to the left* and *to the right* respectively. If these series converge uniformly in any region, they represent analytic solutions of eq. (6) in that region. For example, let $\varphi(x) = (x-a)^{-2}$; then the series for $y_l(x)$ converges uniformly in any finite region of the plane from which the points $a+1$, $a+2$, $\cdots$ have been excluded by small circles drawn about them, and that for $y_r(x)$ in any finite region from which the points $a$, $a-1$, $a-2$, $\cdots$ have been similarly excluded. The solution $y_l(x)$ in this case has poles of the second order at $x = a+1$, $a+2$, $\cdots$, which we will call the points *congruent on the right* to $a$, and $y_r(x)$ has poles of the second order at $x = a$, $a-1$, $a-2$, $\cdots$, or at $a$ and the points *congruent* to it *on the left.*

Further discussion of summation will be found in the next chapter (§ 3); cf. also § 7 of the present chapter.*

## § 3. Fundamental systems of solutions. General solution.

Consider the linear homogeneous difference equation

$$
a_n(x)\,y(x+n) + a_{n-1}(x)\,y(x+n-1) + \cdots + a_0(x)\,y(x) = 0, \tag{11}
$$

whose coefficients $a_i(x)$ are rational (or, more generally, analytic) functions of $x$. If $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$ are

* For a detailed treatment of the subject of summation, see the first half of Nörlund's *Differenzenrechnung.*

particular solutions of this equation, it is easily seen by direct substitution that

$$p_1(x)\,y_1(x) + p_2(x)\,y_2(x) + \cdots + p_n(x)\,y_n(x),$$

where the $p$'s are arbitrary periodic functions of period 1, is also a solution. As previously indicated, we limit ourselves to the case where the $y$'s and the $p$'s are all analytic in the finite part of the plane except for poles. If there exist $n$ periodic functions not all identically zero such that this expression vanishes identically, the solutions $y_1(x), y_2(x), \cdots, y_n(x)$ are said to be *linearly dependent*; otherwise they are *linearly independent* and form a *fundamental system of solutions.*

In determining whether a given set of solutions form a fundamental system or not, the following theorem, due to Casorati, is helpful.

THEOREM. *A necessary and sufficient condition for the linear dependence of the solutions* $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$ *is the identical vanishing of the determinant*

$$D(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1(x+1) & y_2(x+1) & \cdots & y_n(x+1) \\ \cdot & \cdot & \cdot & \cdot \\ y_1(x+n-1) & y_2(x+n-1) & \cdots & y_n(x+n-1) \end{vmatrix}.$$

We will call $D(x)$ the *determinant of Casorati*; it plays the same part for linear difference equations that the Wronskian does for linear differential equations.

To prove the condition necessary, let the $y$'s be linearly dependent; then

$$p_1(x)\,y_1(x) + p_2(x)\,y_2(x) + \cdots + p_n(x)\,y_n(x) = 0$$

identically, and by replacing $x$ by $x+1, x+2, \cdots, x+n-1$ we have also

$$p_1(x)\,y_1(x+1) + p_2(x)\,y_2(x+1) + \cdots + p_n(x)\,y_n(x+1) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_1(x)\,y_1(x+n-1) + p_2(x)\,y_2(x+n-1) + \cdots + p_n(x)\,y_n(x+n-1) = 0.$$

For any particular value of $x$ which is not a pole of any of the periodic functions nor a zero of all of them, these form a set of $n$ linear homogeneous algebraic equations in the $n$ quantities $p_1(x), p_2(x), \cdots, p_n(x)$. Since not all of the latter are zero, it follows that at every such point the determinant of the coefficients, which is $D(x)$, vanishes. But each element of $D(x)$ and hence $D(x)$ itself is an analytic function; consequently, since it vanishes everywhere except possibly at a set of isolated points, it must vanish identically.

Conversely, if $D(x)$ vanishes identically, let $Y_1(x), Y_2(x), \cdots, Y_n(x)$ be the cofactors of the elements of the last row; we will assume for the moment that at least one of them, say $Y_1(x)$, is not identically zero. Then by well known properties of determinants* we have identically

$$(12)\quad \begin{cases} Y_1(x)y_1(x) + Y_2(x)y_2(x) + \cdots + Y_n(x)y_n(x) = 0, \\ Y_1(x)y_1(x+1) + Y_2(x)y_2(x+1) + \cdots \\ \qquad\qquad + Y_n(x)y_n(x+1) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y_1(x)y_1(x+n-1) + Y_2(x)y_2(x+n-1) + \cdots \\ \qquad\qquad + Y_n(x)y_n(x+n-1) = D(x) = 0. \end{cases}$$

The cofactors of the elements of the first row, apart from sign, are $Y_1(x+1), Y_2(x+1), \cdots, Y_n(x+1)$; hence we have similarly

$$(13)\quad \begin{cases} Y_1(x+1)y_1(x) + Y_2(x+1)y_2(x) + \cdots \\ \qquad\qquad + Y_n(x+1)y_n(x) = \pm D(x) = 0, \\ Y_1(x+1)y_1(x+1) + Y_2(x+1)y_2(x+1) + \cdots \\ \qquad\qquad + Y_n(x+1)y_n(x+1) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Y_1(x+1)y_1(x+n-1) + Y_2(x+1)y_2(x+n-1) + \cdots \\ \qquad\qquad + Y_n(x+1)y_n(x+n-1) = 0. \end{cases}$$

Equations (12) form a set of linear homogeneous equations of rank $n-1$ in $Y_1(x), Y_2(x), \cdots, Y_n(x)$, and hence determine uniquely the ratios $Y_2(x)/Y_1(x), \cdots, Y_n(x)/Y_1(x)$; these are analytic except for poles, since they are rational functions

* Cf. e. g. G. Kowalewski: *Determinantentheorie*, § 20.

of the $y$'s. Equations (13) form a similar system in $Y_1(x+1)$, $Y_2(x+1), \cdots, Y_n(x+1)$; since the coefficients are the same, the solutions are the same, so we have

$$\frac{Y_2(x+1)}{Y_1(x+1)} = \frac{Y_2(x)}{Y_1(x)}, \cdots, \frac{Y_n(x+1)}{Y_1(x+1)} = \frac{Y_n(x)}{Y_1(x)};$$

i. e., the ratios are periodic functions. Hence if we divide the first equation in (12) by $Y_1(x)$ and write

$$p_1(x) = 1, \quad p_2(x) = \frac{Y_2(x)}{Y_1(x)}, \quad \cdots, \quad p_n(x) = \frac{Y_n(x)}{Y_1(x)},$$

we have the identical relation

$$p_1(x)y_1(x) + p_2(x)y_2(x) + \cdots + p_n(x)y_n(x) = 0,$$

where $p_1(x)$ at least is not identically zero; hence the solutions $y_1(x), y_2(x), \cdots, y_n(x)$ are linearly dependent.

In case all the $Y$'s vanish identically, we observe that $Y_n(x)$ is the determinant of Casorati for the $n-1$ solutions $y_1(x), y_2(x), \cdots, y_{n-1}(x)$, so it follows by the argument above that these $n-1$ solutions are linearly dependent, and hence all $n$ solutions are linearly dependent, since we can take $p_n(x) = 0$ identically.

The importance of a fundamental system of solutions of eq. (11) is due to the fact that if $y_1(x), y_2(x), \cdots, y_n(x)$ form a fundamental system, *every* solution can be expressed in the form

$$(14) \quad y(x) = p_1(x)y_1(x) + p_2(x)y_2(x) + \cdots + p_n(x)y_n(x)$$

by a suitable choice of the periodic functions. To prove this, let $y(x)$ be an arbitrary solution, and consider the equations

$$(15) \quad \begin{cases} a_n(x)\,y(x+n) + a_{n-1}(x)\,y(x+n-1) + \cdots \\ \qquad\qquad + a_0(x)\,y(x) = 0, \\ a_n(x)\,y_1(x+n) + a_{n-1}(x)\,y_1(x+n-1) + \cdots \\ \qquad\qquad + a_0(x)\,y_1(x) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_n(x)\,y_n(x+n) + a_{n-1}(x)\,y_n(x+n-1) + \cdots \\ \qquad\qquad + a_0(x)\,y_n(x) = 0. \end{cases}$$

which express the fact that $y(x)$, $y_1(x)$, $\cdots$, $y_n(x)$ are solutions of eq. (11). We may without loss of generality assume that not all the coefficients $a_n(x)$, $\cdots$, $a_0(x)$ vanish at the same point, since if they did we could remove the zero by dividing the whole equation by a suitable factor. Hence for every value of $x$ we have

$$\begin{vmatrix} y(x+n) & y(x+n-1) & \cdots & y(x) \\ y_1(x+n) & y_1(x+n-1) & \cdots & y_1(x) \\ \cdot & \cdot & \cdot & \cdot \\ y_n(x+n) & y_n(x+n-1) & \cdots & y_n(x) \end{vmatrix} = 0.$$

from which it follows by Casorati's theorem that a relation of the form

$$p(x)\,y(x) + p_1(x)\,y_1(x) + \cdots + p_n(x)\,y_n(x) = 0$$

holds identically, where $p(x)$ does not vanish identically, since $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$ form a fundamental system; solving this for $y(x)$ gives an equation of the form (14). Equation (14) thus represents *every* solution of eq. (11), and is therefore called the *general solution.*

The particular periodic functions needed to express a definite solution $y(x)$ in terms of a given fundamental system $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$ may be obtained by solving the equations

$$\begin{aligned} &p_1(x)\,y_1(x) + p_2(x)\,y_2(x) + \cdots + p_n(x)\,y_n(x) = y(x), \\ &p_1(x)\,y_1(x+1) + p_2(x)\,y_2(x+1) + \cdots \\ &\qquad\qquad + p_n(x)\,y_n(x+1) = y(x+1), \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ &p_1(x)\,y_1(x+n-1) + p_2(x)\,y_2(x+n-1) + \cdots \\ &\qquad\qquad + p_n(x)\,y_n(x+n-1) = y(x+n-1) \end{aligned}$$

for $p_1(x)$, $p_2(x)$, $\cdots$, $p_n(x)$.

## § 4. Homogeneous equations with constant coefficients.

If the coefficients $a_k(x)$ in eq. (11) are all constants, the equation is easily solved. For simplicity we will take $n = 3$,

but the methods used apply to equations of any order. We consider then the equation

$$(16)\quad a_3\, y(x+3) + a_2\, y(x+2) + a_1\, y(x+1) + a_0\, y(x) = 0,$$

and assume that $a_3 \neq 0$, $a_0 \neq 0$, since otherwise the order is less than 3.

Let us set $y(x) = \varrho^x u(x)$, and seek to determine the constant $\varrho$ and the function $u(x)$ so that this will be a solution of eq. (16). If we express $u(x+1)$, $u(x+2)$, and $u(x+3)$ in terms of the differences of $u(x)$ (cf. § 1), eq. (16) becomes

$$\begin{aligned}\varrho^x (a_3\varrho^3 + a_2\varrho^2 + a_1\varrho + a_0)\, u(x) + \varrho^{x+1}(3a_3\varrho^2 + 2a_2\varrho + a_1)\,\Delta u(x) \\ + \varrho^{x+2}(3a_3\varrho + a_2)\,\Delta^2 u(x) + \varrho^{x+3} a_3\,\Delta^3 u(x) = 0,\end{aligned}$$

or, if we cancel the factor $\varrho^x$ and set

$$f(\varrho) = a_3\varrho^3 + a_2\varrho^2 + a_1\varrho + a_0,$$

$$(17)\quad \begin{aligned} f(\varrho)\, u(x) + \varrho f'(\varrho)\,\Delta u(x) + \varrho^2 \frac{f''(\varrho)}{2!}\,\Delta^2 u(x) \\ + \varrho^3 \frac{f'''(\varrho)}{3!}\,\Delta^3 u(x) = 0.\end{aligned}$$

Let $\varrho_1$, $\varrho_2$, and $\varrho_3$ be the roots of the *characteristic equation* $f(\varrho) = 0$. If we set $\varrho$ equal to one of these and $u(x)$ equal to a constant, eq. (17) will be satisfied, since the difference of a constant is zero; hence particular solutions of eq. (16) are $\varrho_1^x$, $\varrho_2^x$, $\varrho_3^x$. If the roots of the characteristic equation are distinct, these form a fundamental system, for

$$D(x) = (\varrho_1\varrho_2\varrho_3)^x (\varrho_1 - \varrho_2)(\varrho_2 - \varrho_3)(\varrho_3 - \varrho_1),$$

which never vanishes. The general solution is therefore

$$\varrho_1^x p_1(x) + \varrho_2^x p_2(x) + \varrho_3^x p_3(x),$$

where the $p$'s are arbitrary periodic functions of period 1. If the characteristic equation has two equal roots, say $\varrho_1 = \varrho_2$, then $f'(\varrho_1) = 0$, and for $\varrho = \varrho_1$ eq. (17) becomes

$$\varrho_1^2 \frac{f''(\varrho_1)}{2!} \Delta^2 u(x) + \varrho_1^3 \frac{f'''(\varrho_1)}{3!} \Delta^3 u(x) = 0.$$

which is satisfied if $u(x)$ is a constant or a polynomial of the first degree. Particular solutions are therefore $\varrho_1^x$, $x\varrho_1^x$, $\varrho_3^x$, and the general solution is

$$\varrho_1^x [p_1(x) + x p_2(x)] + \varrho_3^x.$$

for $D(x) = \varrho_1^{2x+1} \varrho_3^x (\varrho_1 - \varrho_3)^2$, which never vanishes.

If all three roots of the characteristic equation are equal, then $f'(\varrho_1) = f''(\varrho_1) = 0$, and all the terms of eq. (17) vanish except the last; it is satisfied if $u(x)$ is any polynomial of degree not greater than 2. Particular solutions are $\varrho_1^x$, $x\varrho_1^x$, $x^2\varrho_1^x$, and the general solution is

$$\varrho_1^x [p_1(x) + x p_2(x) - x^2 p_3(x)]:$$

in this case $D(x) = 2\varrho_1^{3x+3}$.

Difference equations with constant coefficients have many applications in other branches of mathematics, both pure and applied. The recursion formulas, for instance, by which each term of certain series is obtained from a number of preceding terms, can be regarded as equations of this sort. A simple example is furnished by the *numbers of Fibonacci*:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots,$$

in which each term after the second is the sum of the two preceding terms. If we call the terms $y(0)$, $y(1)$, $y(2)$, $\cdots$, we have for positive integral values of $x$

$$y(x+2) - y(x+1) - y(x) = 0.$$

The roots of the characteristic equation

$$\varrho^2 - \varrho - 1 = 0$$

are $\frac{1}{2}(1 \pm \sqrt{5})$, and the general solution is therefore

$$y(x) = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^x + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^x,$$

where $c_1$ and $c_2$ may be considered as constants instead of periodic functions, since $x$ is limited to integral values. The values of $c_1$ and $c_2$ which give the numbers of Fibonacci are obtained from the fact that $y(0) = 0$ and $y(1) = 1$, namely $c_1 = 1/\sqrt{5}$, $c_2 = -1/\sqrt{5}$. The general term of the series is therefore

$$y(x) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1-\sqrt{5}}{2}\right)^x\right].$$

## § 5. Non-homogeneous equations.

Consider the non-homogeneous linear difference equation

$$(18)\qquad \begin{aligned} a_n(x)\,y(x+n) + a_{n-1}(x)\,y(x+n-1) + \cdots + a_0(x)\,y(x) \\ = b(x). \end{aligned}$$

The equation obtained from this by setting $b(x) = 0$ is called the *associated homogeneous equation.* Solutions of eq. (18) can be derived from those of the associated homogeneous equation by a method analogous to that of "variation of constants" used in solving a non-homogeneous linear differential equation.

Let $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$ be a fundamental system of solutions of the associated homogeneous equation, and set

$$(19)\quad y(x) = c_1(x)\,y_1(x) + c_2(x)\,y_2(x) + \cdots + c_n(x)\,y_n(x);$$

we seek to determine the functions $c_1(x)$, $\cdots$, $c_n(x)$ so that $y(x)$ shall be a solution of eq. (18). Replacing $x$ by $x+1$, and writing $c_k(x+1) = c_k(x) + \Delta c_k(x)$, we have

$$\begin{aligned} y(x+1) = {} & c_1(x)\,y_1(x+1) + c_2(x)\,y_2(x+1) + \cdots \\ & + c_n(x)\,y_n(x+1) \\ & + [\Delta c_1(x)\,y_1(x+1) + \Delta c_2(x)\,y_2(x+1) + \cdots \\ & + \Delta c_n(x)\,y_n(x+1)]. \end{aligned}$$

Suppose the $c$'s are so chosen that the expression in brackets vanishes identically. Replacing $x$ by $x+1$ again, we have

$$\begin{aligned} y(x+2) = \quad & c_1(x)\,y_1(x+2) + \quad c_2(x)\,y_2(x+2) + \cdots \\ & \qquad\qquad\qquad\qquad + \quad c_n(x)\,y_n(x+2) \\ & + [\Delta c_1(x)\,y_1(x+2) + \Delta c_2(x)\,y_2(x+2) + \cdots \\ & \qquad\qquad\qquad\qquad + \Delta c_n(x)\,y_n(x+2)], \end{aligned}$$

and again we assume that the expression in brackets vanishes identically. Continuing in this way, we have finally

$$\begin{aligned} y(x+n) = \quad & c_1(x)\,y_1(x+n) + \quad c_2(x)\,y_2(x+n) + \cdots \\ & \qquad\qquad\qquad\qquad + \quad c_n(x)\,y_n(x+n) \\ & + [\Delta c_1(x)\,y_1(x+n) + \Delta c_2(x)\,y_2(x+n) + \cdots \\ & \qquad\qquad\qquad\qquad + \Delta c_n(x)\,y_n(x+n)], \end{aligned}$$

and in this last equation we will retain the terms in brackets. Multiplying these $n+1$ equations by $a_0(x)$, $a_1(x)$, $\cdots$, $a_n(x)$ respectively and adding, remembering that $y_1(x)$, $\cdots$, $y_n(x)$ are solutions of the associated homogeneous equation, we obtain the equation

$$\begin{aligned} b(x) = a_n(x)\,[\Delta c_1(x)\,y_1(x+n) + \Delta c_2(x)\,y_2(x+n) + \cdots \\ + \Delta c_n(x)\,y_n(x+n)]. \end{aligned}$$

To determine the $\Delta c$'s we now have the following $n$ equations:

$$\begin{aligned} &\Delta c_1(x)\,y_1(x+1) &&+ \cdots + \Delta c_n(x)\,y_n(x+1) &&= 0, \\ &\cdots\cdots\cdots\cdots &&\cdots\cdots\cdots\cdots && \\ &\Delta c_1(x)\,y_1(x+n-1) &&+ \cdots + \Delta c_n(x)\,y_n(x+n-1) &&= 0, \\ &\Delta c_1(x)\,y_1(x+n) &&+ \cdots + \Delta c_n(x)\,y_n(x+n) &&= \frac{b(x)}{a_n(x)}. \end{aligned}$$

The determinant of the coefficients is $D(x+1)$, which does not vanish identically, since the $y$'s form a fundamental system: hence these equations determine uniquely the differences $\Delta c_1(x)$, $\cdots$, $\Delta c_n(x)$. In fact,

$$\Delta c_k(x) = \frac{M_{nk}(x)}{D(x+1)}\,\frac{b(x)}{a_n(x)} \qquad (k = 1, 2, \cdots, n),$$

where $M_{nk}(x)$ is the cofactor of the last element in the $k$th column of $D(x+1)$*. The $c$'s themselves can be obtained

* If $n=1$, $M_{nk}(x)$ is to be replaced by 1.

by summation. Putting in (19) the values thus obtained, we have

$$(20) \qquad y(x) = \sum_{k=1}^{n} y_k(x) \mathop{\mathsf{S}} \frac{M_{nk}(x)}{D(x+1)} \frac{b(x)}{a_n(x)}$$

as a solution of eq. (18).

As an example of this procedure, consider the equation

$$y(x+2) - 7y(x+1) + 6y(x) = x.$$

The associated homogeneous equation has the linearly independent solutions $1$, $6^x$, for which

$$D(x+1) = \begin{vmatrix} 1 & 6^{x+1} \\ 1 & 6^{x+2} \end{vmatrix} = 30 \cdot 6^x,$$

and $M_{21}(x) = -6^{x+1}$, $M_{22}(x) = 1$. Hence

$$\begin{aligned} y(x) &= \mathop{\mathsf{S}} \frac{-x6^{x+1}}{30 \cdot 6^x} + 6^x \mathop{\mathsf{S}} \frac{x}{30 \cdot 6^x} \\ &= -\frac{1}{5} \mathop{\mathsf{S}} x + \frac{6^x}{30} \mathop{\mathsf{S}} \frac{x}{6^x} \\ &= -\frac{1}{5} \frac{x(x-1)}{2} + \frac{6^x}{30}\left(-\frac{30x+6}{25 \cdot 6^x}\right) \\ &= -\frac{x^2}{10} + \frac{3x}{50} - \frac{1}{125}. \end{aligned}$$

If $\eta_1(x)$ and $\eta_2(x)$ are any two solutions of eq. (18), we see by substituting them in the equation and subtracting that their difference is a solution of the associated homogeneous equation. Hence the most general solution is obtained by adding to any particular solution the general solution of the associated homogeneous equation; this gives

$$(21) \qquad y(x) = \eta(x) + p_1(x)\, y_1(x) + \cdots + p_n(x)\, y_n(x),$$

where $\eta(x)$ is a particular solution of eq.(18) and $p_1(x), \cdots, p_n(x)$ are arbitrary periodic functions. Thus the most general solution of the example above is

$$p_1(x) + 6^x p_2(x) - \frac{x^2}{10} + \frac{3x}{50} - \frac{1}{125}.$$

Another method of handling a non-homogeneous equation is to reduce it to a homogeneous equation of higher order. If we replace $x$ by $x+1$ in eq. (18) and multiply by $b(x)$, we have

$$b(x)\,[a_n(x+1)\,y(x+n+1) + \cdots + a_0(x+1)\,y(x+1)] \\ = b(x)\,b(x+1);$$

if we multiply eq. (18) by $b(x+1)$ we have

$$b(x+1)\,[a_n(x)\,y(x+n) + \cdots + a_0(x)\,y(x)] \\ = b(x)\,b(x+1).$$

Subtracting one of these equations from the other gives us a homogeneous equation of order $n+1$, which must be satisfied by every solution $y(x)$ of eq. (18). It is also satisfied by every solution of the associated homogeneous equation, since these make the expressions in brackets in the last two equations vanish. Thus we obtain for our numerical example the third order equation

$$xy(x+3) - (8x+1)\,y(x+2) + (13x+7)\,y(x+1) \\ - 6(x+1)\,y(x) = 0.$$

which has the linearly independent solutions 1, $6^x$, and $-\frac{x^2}{10} + \frac{3x}{50} - \frac{1}{125}$.

## § 6. Asymptotic representation.

The idea of asymptotic representation plays an important part in the theory of linear difference equations. For our purposes it may be defined as follows. Let $f(x)$ be a function of the complex variable $x$ defined on a ray which starts at some finite point $\alpha$ and goes to $\infty$, and let

$$S(x) = s_0 + \frac{s_1}{x} + \frac{s_2}{x^2} + \cdots$$

be a series in $1/x$, convergent or divergent; then if there exist, for every $n$, positive constants $M$ and $R$ such that

$$\left| f(x) - \left(s_0 + \frac{s_1}{x} + \cdots + \frac{s_n}{x^n}\right)\right| < \frac{M}{|x|^n}$$

when $|x| > R$ on the ray, $f(x)$ is said to *be represented asymptotically* by $S(x)$ [in symbols, $f(x) \sim S(x)$] along the ray. The constants $M$ and $R$ in general depend on $n$. More generally, if $f(x)$ is defined in the sector $\theta_1 < \arg(x-\alpha) < \theta_2$ of the complex plane, we will say that $f(x)$ is represented asymptotically by $S(x)$ in this sector if the above inequality holds when $|x| > R$ in the closed sector $\theta_1 + \varepsilon \leqq \arg(x-\alpha) \leqq \theta_2 - \varepsilon$, where $\varepsilon$ is an arbitrarily small positive angle. In this case $M$ and $R$ depend on $\varepsilon$ as well as on $n$. If $f(x)$ is also defined on the boundaries of the sector, and if the inequality holds when $|x| > R$ in the closed sector $\theta_1 \leqq \arg(x-\alpha) \leqq \theta_2$, we will say that $f(x)$ is represented asymptotically by $S(x)$ in the latter sector.

The definition may also be stated in the form of an equality, namely

$$f(x) = s_0 + \frac{s_1}{x} + \cdots + \frac{s_n}{x^n} + \frac{\sigma_n(x)}{x^n},$$

where $|\sigma_n(x)| < M$ when $|x| > R$. If we replace $n$ by $n+1$, and set

$$\frac{s_{n+1}}{x} + \frac{\sigma_{n+1}(x)}{x} = \varepsilon_n(x),$$

we may write

(22) $$f(x) = s_0 + \frac{s_1}{x} + \cdots + \frac{s_n}{x^n} + \frac{\varepsilon_n(x)}{x^n},$$

where $\lim\limits_{x \to \infty} \varepsilon_n(x) = 0$. Still another form for the definition is

$$\lim_{x \to \infty} x^n \left[ f(x) - \left(s_0 + \frac{s_1}{x} + \cdots + \frac{s_n}{x^n}\right)\right] = 0.$$

As a simple example,* consider the function

$$f(x) = \int_x^\infty \frac{e^{x-t}}{t}\, dt,$$

* Whittaker and Watson: *Modern Analysis* (3rd ed.), p. 150.

where $x$ is real and positive and the path of integration is the positive axis of reals. Repeated integration by parts gives

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n} + (-1)^n n! \int_x^{\infty} \frac{e^{x-t}}{t^{n+1}}\, dt.$$

We are thus led to consider the infinite series

$$S(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots + \frac{(-1)^n n!}{x^{n+1}} - \cdots.$$

This is obviously divergent for all values of $x$; however, if we let $S_n(x)$ denote the sum of the first $n$ terms of $S(x)$, we have

$$|f(x) - S_n(x)| = n! \int_x^{\infty} \frac{e^{x-t}}{t^{n+1}}\, dt$$

$$< n! \int_x^{\infty} \frac{dt}{t^{n+1}} = \frac{(n-1)!}{x^n}.$$

Accordingly, by the definition above, $f(x) \sim S(x)$ along the positive axis of reals.

The value of $f(x)$ for large values of $x$ can be calculated with great accuracy from the asymptotic series. In this way a divergent series may give almost as much information about the function which it represents as a convergent series would do. In fact, in cases where both convergent and divergent series representing a function are known, the latter is often preferred in numerical computation because fewer terms are required to obtain the value of the function with the desired degree of accuracy.

The laws of operation for asymptotic series are essentially the same as those for convergent series.* The most important for our purposes are the following. Let $f(x)$ and $g(x)$ be two functions of $x$, and

* For a general treatment of asymptotic series see Ford: *Studies on Divergent Series and Summability*, Chaps. I–III.

$$S(x) = s_0 + \frac{s_1}{x} + \frac{s_2}{x^2} + \dots,$$

$$T(x) = t_0 + \frac{t_1}{x} + \frac{t_2}{x^2} + \dots$$

two infinite series, convergent or divergent. Let $S_n(x)$ and $T_n(x)$ denote the sums of the first $n+1$ terms of these series.

I. *If $f(x) \sim S(x)$ and $g(x) \sim T(x)$ in any sector, then $f(x) \pm g(x) \sim S(x) \pm T(x)$.*

For, by eq. (22),

$$(23)\quad f(x) = S_n(x) + \frac{\varepsilon_n(x)}{x^n},\ g(x) = T_n(x) + \frac{\varepsilon'_n(x)}{x^n};$$

adding and subtracting, we have

$$f(x) \pm g(x) = S_n(x) \pm T_n(x) + \frac{\varepsilon_n(x) \pm \varepsilon'_n(x)}{x^n}.$$

Since both $\varepsilon_n(x)$ and $\varepsilon'_n(x)$ approach zero as $x \to \infty$, the same is true of their sum and difference; hence $f(x) \pm g(x) \sim S(x) \pm T(x)$.

II. *If $f(x) \sim S(x)$ and $g(x) \sim T(x)$, then $f(x)\,g(x) \sim S(x)\,T(x)$.*

For multiplying eqs. (23) together, we have

$$f(x)\,g(x) = s_0\,t_0 + \frac{s_0\,t_1 + s_1\,t_0}{x} + \dots + \frac{s_0\,t_n + \dots + s_n\,t_0}{x^n} + \frac{S_n(x)\,\varepsilon'_n(x) + T_n(x)\,\varepsilon_n(x) + \eta(x)}{x^n},$$

where $\lim\limits_{x \to \infty} \eta(x) = 0$. Since $S_n(x)$ and $T_n(x)$ approach the finite limits $s_0$ and $t_0$ respectively as $x \to \infty$, the numerator of the last term approaches zero. The other terms on the right consist of the first $n+1$ terms of the product $S(x)\,T(x)$; therefore, by (22), $f(x)\,g(x) \sim S(x)\,T(x)$.

III. *If $f(x) \sim S(x)$ and $g(x) \sim T(x)$, and if $t_0 \neq 0$, then $f(x)/g(x) \sim S(x)/T(x)$.*

If we divide 1 by the series $T(x)$, we obtain formally a series

$$\frac{1}{T(x)} = u_0 + \frac{u_1}{x} + \frac{u_2}{x^2} + \dots;$$

since

$$g(x) = T_n(x) + \frac{\varepsilon'_n(x)}{x^n}, \tag{24}$$

$1/g(x)$ is equal to $1/T(x)$ to terms of the $n$th degree, so we may write

$$\frac{1}{g(x)} = U_n(x) + \frac{\varrho_n(x)}{x^n}, \tag{25}$$

where $U_n(x)$ is the sum of the first $n+1$ terms of the series $1/T(x)$. If we multiply eqs. (24) and (25), we have

$$1 = U_n(x)\left[T_n(x) + \frac{\varepsilon'_n(x)}{x^n}\right] + \frac{g(x)\,\varrho_n(x)}{x^n}.$$

We may write

$$U_n(x)\,T_n(x) = 1 + \frac{\eta(x)}{x^n}.$$

where $\lim\limits_{x\to\infty} \eta(x) = 0$, since $U_n(x)$ and $T_n(x)$ are the first $n+1$ terms of the reciprocals $1/T(x)$ and $T(x)$; hence

$$\eta(x) + U_n(x)\,\varepsilon'_n(x) + g(x)\,\varrho_n(x) = 0.$$

$$\varrho_n(x) = -\frac{\eta(x) + U_n(x)\,\varepsilon'_n(x)}{g(x)}.$$

Since the numerator approaches zero and the denominator approaches the value $t_0 \neq 0$ as $x\to\infty$, we see that $\lim\limits_{n\to\infty} \varrho_n(x) = 0$. It now follows from eq. (25) that $1/g(x) \sim 1/T(x)$, and consequently, by II, that $f(x)/g(x) \sim S(x)/T(x)$.

IV. *If $f(x)\sim S(x)$ in the sector $\theta_1 < \arg(x-\alpha) < \theta_2$, then $\frac{d}{dx} f(x) \sim \frac{d}{dx} S(x)$ in the sector $\theta_1 < \arg(x-\alpha') < \theta_2$, where $\alpha'$ is any point in the first sector.*

The second sector lies within the first, and is bounded by two rays parallel to those bounding the first sector. Let $d$ be the distance between the closer of these two pairs of parallel rays; a circle $C$ of radius $d$ with center at any point $x$ of the inner sector will then lie wholly in the outer sector. Differentiating eq. (22), we have

$$\frac{d}{dx} f(x) = -\frac{s_1}{x^2} - \frac{2 s_2}{x^3} - \cdots$$
$$- \frac{n s_n}{x^{n+1}} - \frac{n \varepsilon_n(x)}{x^{n+1}} + \frac{1}{x^n} \frac{d}{dx} \varepsilon_n(x).$$

By Cauchy's integral formula,

$$\frac{d}{dx} \varepsilon_n(x) = \frac{1}{2\pi i} \int_C \frac{\varepsilon_n(t)\, dt}{(t-x)^2},$$

where the integration is around the circle $C$ in the positive direction. Since $\lim\limits_{x\to\infty} \varepsilon_n(x) = 0$, we have $|\varepsilon_n(x)| < \varepsilon$ for $|x| > R$, where $\varepsilon$ is an arbitrarily small positive constant. If we take $|x| > R + d$, then for all points of $C$ we have $|\varepsilon_n(t)| < \varepsilon$, so that

$$\left| \frac{d}{dx} \varepsilon_n(x) \right| < \frac{1}{2\pi} \int_0^{2\pi} \frac{\varepsilon\, d\theta}{d} = \frac{\varepsilon}{d} = \varepsilon',$$

where $\varepsilon'$ is arbitrarily small. Hence

$$\frac{d}{dx} f(x) = \frac{d}{dx} S_n(x) + \frac{\eta(x)}{x^n},$$

where

$$\lim_{x\to\infty} \eta(x) = \lim_{x\to\infty} \left[ \frac{d}{dx} \varepsilon_n(x) - \frac{n \varepsilon_n(x)}{x} \right] = 0$$

in the inner sector; i. e., $\frac{d}{dx} f(x) \sim \frac{d}{dx} S(x)$ in this sector.

Let

$$\varphi(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

be a function analytic in the neighborhood of $x = 0$; the series will then be convergent inside of some circle $|x| = r$. Let

$$S(x) = \frac{s_1}{x} + \frac{s_2}{x^2} + \cdots$$

be a series in $1/x$ without a constant term, convergent or divergent.

V. *If $f(x) \sim S(x)$ in a sector, then*

$$\varphi(f(x)) \sim c_0 + c_1 S(x) + c_2 [S(x)]^2 + \cdots,$$

*where the last series is to be arranged in powers of* $1/x$.

Since the series for $\varphi(x)$ converges when $|x| < r$,

$$\varphi(f(x)) = c_0 + c_1 f(x) + \cdots + c_n [f(x)]^n + \mu(x) [f(x)]^{n+1},$$

where $\mu(x)$ is bounded, provided $|f(x)| < r$. But $|f(x)| < \frac{M}{|x|}$ when $|x| > R$ in the sector, so evidently $|f(x)| < r$ for sufficiently large values of $x$. For such values

$$|\varphi(f(x)) - c_0 - c_1 f(x) - \cdots - c_n [f(x)]^n| < |\mu(x)| \frac{M^{n+1}}{|x^{n+1}|}.$$

If we write

$$f(x) = \frac{s_1}{x} + \cdots + \frac{s_n}{x^n} + \frac{\sigma_n(x)}{x^n} = S_n(x) + \frac{\sigma_n(x)}{x^n}$$

and expand the powers of $f(x)$, arranging the terms in powers of $1/x$, it is clear that the left-hand side of the last inequality differs from

$$\varphi(f(x)) - c_0 - c_1 S_n(x) - \cdots - c_n [S_n(x)]^n$$

only by terms of degree $n$ or greater in $1/x$. Hence

$$|\varphi(f(x)) - c_0 - c_1 S_n(x) - \cdots - c_n [S_n(x)]^n| < \frac{M'}{|x^n|}$$

for sufficiently large values of $x$ in the sector; i. e.,

$$\varphi(f(x)) \sim c_0 + c_1 S(x) + c_2 [S(x)]^2 + \cdots.$$

VI. *A function cannot be represented asymptotically by two distinct series in the same sector.*

For if

$$f(x) \sim s_0 + \frac{s_1}{x} + \frac{s_2}{x^2} + \cdots$$

and

$$f(x) \sim t_0 + \frac{t_1}{x} + \frac{t_2}{x^2} + \cdots,$$

we have by subtraction

$$0 \sim s_0 - t_0 + \frac{s_1 - t_1}{x} + \frac{s_2 - t_2}{x^2} + \cdots;$$

but this cannot be true unless $s_0 - t_0 = 0$, or $s_0 = t_0$; multiplying by $x$, we see that we must also have $s_1 = t_1$, etc.

An interesting consequence of this uniqueness of asymptotic representation is the fact that an analytic function $f(x)$ cannot be represented asymptotically by a *divergent* series $S(x)$ in the *complete* neighborhood of $x = \infty$. For if it could be so represented, it would be bounded in this neighborhood and equal to $s_0$ at the point $x = \infty$, and hence would be analytic there. Its expansion in a convergent power series would then, by VI, necessarily be identical with the asymptotic series, which contradicts the assumption that the latter is divergent.

We shall find it convenient to extend our use of the symbol for asymptotic representation to the following cases. (a) If $f(x) - a(x) \sim S(x)$, where $a(x)$ is any function, we write $f(x) \sim a(x) + S(x)$. (b) If $f(x)/a(x) \sim S(x)$, and if $a(x)$ does not vanish for $|x| > R$, we write $f(x) \sim a(x)\, S(x)$. (c) If $f(x) = f_1(x) + f_2(x)$, where $f_1(x) \sim S_1(x)$ and $f_2(x) \sim S_2(x)$, we write $f(x) \sim S_1(x) + S_2(x)$.

In connection with (b) it should be noted that the meanings of the relations $f(x) \sim a(x) \cdot 0$ and $f(x) \sim 0$ are entirely different; the former means that $\lim\limits_{x \to \infty} x^n \left| \frac{f(x)}{a(x)} \right| = 0$, and the latter that $\lim\limits_{x \to \infty} x^n |f(x)| = 0$. For example, in the right half plane we have $1 \sim 1$ and at the same time $1 \sim e^x \cdot 0$.

Sometimes the statement "$f(x)$ is asymptotic to $S(x)$" is used for brevity instead of "$f(x)$ is represented asymptotically by $S(x)$".

## § 7. The function $\Phi(x, r, k)$.

As an application of the ideas of § 6, we will study the function*

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* This is an example of an important class of series of the form $\Sigma c_n\, g(x + n)$. Cf. Carmichael, *Trans. Amer. Math. Soc.*, 17 (1916), pp. 207–232; *Bull. Amer. Math. Soc.*, (2), 23 (1917), pp. 407–425.

$$(26)\qquad \Phi(x, r, k) = \frac{1}{x^k} + \frac{r}{(x+1)^k} + \frac{r^2}{(x+2)^k} + \cdots,$$

where $k$ is an integer and $|r| \leqq 1$. It may readily be verified that $\Phi$ is a solution of the difference equation

$$(27)\qquad ry(x+1) - y(x) = -\frac{1}{x^k}.$$

In this study we need the following lemma, which will also be useful later.

LEMMA.* *If $k$ is a positive integer $\geqq 2$, then*

$$(28)\qquad \sum_{n=0}^{\infty} \frac{1}{|x+n|^k} < \frac{1}{|x|^{k-1}}\left(\frac{1}{|x|} + \frac{\pi}{2}\right)$$

*when $x$ is in the right half plane, and*

$$(29)\qquad \sum_{n=0}^{\infty} \frac{1}{|x+n|^k} < \frac{2}{|v|^{k-1}}\left(\frac{1}{|v|} + \frac{\pi}{2}\right)$$

*when $x$ is in the left half plane.*

When $x$ lies in the right half plane $u$ is positive, so that

$$|x+n|^2 = (u+n)^2 + v^2 > u^2 + v^2 + n^2 = |x|^2 + n^2,$$

whence

$$\sum_{n=0}^{\infty} \frac{1}{|x+n|^k} < \sum_{n=0}^{\infty} \frac{1}{(|x|^2+n^2)^{k/2}} < \frac{1}{|x|^{k-2}} \sum_{n=0}^{\infty} \frac{1}{(|x|^2+n^2)}$$

$$< \frac{1}{|x|^{k-2}}\left[\frac{1}{|x|^2} + \int_0^{\infty} \frac{dn}{|x|^2+n^2}\right]$$

$$= \frac{1}{|x|^{k-1}}\left(\frac{1}{|x|} + \frac{\pi}{2}\right).$$

When $x$ lies in the left half plane, $u$ is negative, and the series may be written

$$\sum_{n=0}^{\infty} \frac{1}{[(u+n)^2+v^2]^{k/2}} < \sum_{n=-\infty}^{\infty} \frac{1}{[(u+n)^2+v^2]^{k/2}}$$

$$< \frac{1}{|v|^{k-2}} \sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2+v^2}.$$

* Cf. Birkhoff, *Trans. Amer. Math. Soc.*, 12 (1911), p. 248.

the last sum is not decreased if we replace $|u+n|$ by the largest integer which does not exceed it, so we have

$$\sum_{n=0}^{\infty} \frac{1}{|x+n|^k} \leqq \frac{2}{|v|^{k-2}} \sum_{n=0}^{\infty} \frac{1}{n^2+v^2} < \frac{2}{|v|^{k-2}} \left( \frac{1}{v^2} + \int_0^{\infty} \frac{dn}{n^2+v^2} \right)$$

$$= \frac{2}{|v|^{k-1}} \left( \frac{1}{|v|} + \frac{\pi}{2} \right).$$

A similar argument shows that

$$(30) \qquad \sum_{n=0}^{\infty} \frac{1}{|x-n|^k} < \frac{2}{|v|^{k-1}} \left( \frac{1}{|v|} + \frac{\pi}{2} \right)$$

when $x$ is in the right half plane and

$$(31) \qquad \sum_{n=0}^{\infty} \frac{1}{|x-n|^k} < \frac{1}{|x|^{k-1}} \left( \frac{1}{|x|} + \frac{\pi}{2} \right)$$

when $x$ is in the left half plane.

THEOREM. *If $|r| < 1$, the function $\Phi(x, r, k)$ is single-valued and analytic throughout the x-plane when $k < 0$, but has poles of the kth order at $x = 0, -1, -2, \cdots$ when $k > 0$. It satisfies the inequality*

$$(32) \qquad |\Phi(x, r, k)| < \frac{M}{|x|^k} \qquad (M \text{ a constant})$$

*at all points $x$ sufficiently far from the negative axis of reals. If $r = 1$ and $k \geqq 2$, $\Phi(x, r, k)$ is analytic except for poles of the kth order at $x = 0, -1, -2, \cdots$, and satisfies the inequalities*

$$(33) \qquad |\Phi(x, r, k)| < \frac{M}{|x|^{k-1}}, \qquad |\Phi(x, r, k)| < \frac{M}{|v|^{k-1}}$$

*in the right and in the left half planes respectively, except close to the negative axis of reals.*

The ratio of the $(n+1)$st term to the $n$th is $r\left(\frac{x+n-1}{x+n}\right)^k$. If $x$ lies in the right half plane, we have for sufficiently large values of $n$

$$(34) \qquad \left|\frac{x+n-1}{x+n}\right|^k < 1+\varepsilon,$$

where $\varepsilon$ is arbitrarily small. If $|r|<1$, we can choose $\varepsilon$ so small that $|r|(1+\varepsilon) = \varrho < 1$. Then from a certain point on the series is less term by term in absolute value than the convergent series

$$(35) \qquad \frac{1}{|x|^k}(1+\varrho+\varrho^2+\cdots);$$

hence $\Phi(x, r, k)$ converges uniformly in the neighborhood of any point in the right half plane, and so represents there a single-valued analytic function. Moreover, if $|x|$ is sufficiently large, say $|x|>R$, (34) will be true for *all* values of $n$, and hence every term of $\Phi(x, r, k)$ (except the first) will be less in absolute value than the corresponding term of (35). Hence, when $|x|>R$ in the right half plane, the inequality (32) is satisfied, where

$$M = 1+\varrho+\varrho^2+\cdots = \frac{1}{1-\varrho}.$$

If $x$ lies in the left half plane, an integer $\lambda$ can be chosen so large that $x+\lambda = x_1$ is in the right half plane. The part of the series after the $\lambda$th term may be written

$$r^\lambda\left[\frac{1}{x_1^k}+\frac{r}{(x_1+1)^k}+\frac{r^2}{(x_1+2)^k}+\cdots\right].$$

and the argument above shows that this converges uniformly and represents a single-valued analytic function. The sum of the first $\lambda$ terms is a rational function which has poles of the $k$th order at $x = 0, -1, -2, \cdots$ if $k>0$. Hence $\Phi(x, r, k)$ is single-valued and analytic in the entire finite plane when $|r|<1$, except for poles of the $k$th order at $x = 0, -1, -2, \cdots$ when $k>0$. If $x$ is any point in the left half plane for which $|x|>R$, then $|x+n|>R$ for all values of $n$, and as before the inequality (32) is satisfied.

If $r = 1$, we have

$$\Phi(x, 1, k) = \frac{1}{x^k} + \frac{1}{(x+1)^k} + \frac{1}{(x+2)^k} + \cdots.$$

If $k \leq 1$ this series diverges, but if $k \geq 2$ it converges uniformly in the neighborhood of every point except $x = 0$, $-1, -2, \cdots$, where it has poles of the $k$th order. Replacing the terms of the series by their absolute values, we have

$$|\Phi(x, 1, k)| \leq \sum_{n=0}^{\infty} \frac{1}{|x+n|^k},$$

and it follows from (28) and (29) that the inequalities (33) are satisfied; if we exclude the part of the plane within the distance $\varepsilon$ of the negative axis of reals, we can take $M = 2/\varepsilon + \pi$. This completes the proof of our theorem.

When $r = 1$, eq. (27) is of the form (6), so $\Phi(x, 1, k)$ is the sum of the function $-1/x^k$. It is the symbolic solution of (27) to the right; the symbolic solution to the left is $(-1)^{k+1}\Phi(-x, 1, k) + 1/x^k$.

We will now obtain the asymptotic form of $\Phi(x, r, k)$ for large values of $x$, under the condition that $|r| < 1$. (The case $r = 1$ is considered in the next chapter.) Let $x$ have any definite finite value, and consider the first $m+1$ terms of $\Phi$, where $m$ is the greatest integer less than $\frac{1}{2}|x|$. In any of these terms we can write

$$(36)\qquad \begin{aligned} \frac{1}{(x+j)^k} &= \frac{1}{x^k}\left(1+\frac{j}{x}\right)^{-k} \\ &= \frac{1}{x^k}\left[1 - k\frac{j}{x} + \cdots + \binom{-k}{n}\left(\frac{j}{x}\right)^n + \left(\frac{j}{x}\right)^n \theta\left(\frac{j}{x}\right)\right], \end{aligned}$$

where $n$ is any positive integer, and $\theta(j/x)$ is bounded, say $|\theta(j/x)| < M_1$ $(j = 0, 1, 2, \cdots, m)$. The sum of these $m+1$ terms is thus

$$(37)\qquad \begin{aligned} &\frac{1}{x^k}\sum_{j=0}^{m} r^j\left[1 - k\frac{j}{x} + \cdots + \binom{-k}{n}\left(\frac{j}{x}\right)^n\right] \\ &\qquad + \frac{1}{x^k}\sum_{j=0}^{m} r^j\left(\frac{j}{x}\right)^n \theta\left(\frac{j}{x}\right). \end{aligned}$$

But we have

$$a_0 = \sum_{j=0}^{m} r^j = \frac{1}{1-r} - \frac{r^{m+1}}{1-r},$$

$$a_1 = \sum_{j=0}^{m} j\,r^j = r\frac{d\,a_0}{d\,r}$$

$$= \frac{r}{(1-r)^2} - r^{m+1}\left[\frac{m+1}{1-r} + \frac{r}{(1-r)^2}\right].$$

$$a_2 = \sum_{j=0}^{m} j^2\,r^j = r\frac{d\,a_1}{d\,r}, \text{ etc.}$$

Hence, since $m+1 \geqq \frac{1}{2}|x|$ and $|r| < 1$,

$$\left|a_0 - \frac{1}{1-r}\right| = \frac{|r|^{m+1}}{|1-r|} < \frac{|r|^{\frac{1}{2}|x|}}{|1-r|}.$$

Since $|r|^{\frac{1}{2}|x|}$ approaches 0 more rapidly as $x$ increases than any power of $x$, we may write

$$a_0 = \frac{1}{1-r} + \frac{\tau_0(x)}{x^n},$$

where $\tau_0(x)$ is bounded for large values of $x$. Similarly we have

$$a_1 = \frac{r}{(1-r)^2} + \frac{\tau_1(x)}{x^n},$$

$$a_2 = \frac{r+r^2}{(1-r)^3} + \frac{\tau_2(x)}{x^n}, \text{ etc.},$$

where the $\tau$'s are all bounded for large values of $x$ on account of the factor $r^{m+1}$ which appears in each one. If we let $\tau'(x)/x^n$ denote the last sum in (37), we have, since $\left|\theta\left(\frac{j}{x}\right)\right| < M_1$,

$$\left|\frac{\tau'(x)}{x^n}\right| < \frac{M_1}{|x|^n}\sum_{j=0}^{\infty}|r|^j j^n = \frac{M_1 M_2}{|x|^n},$$

where $M_2$ is the sum of the convergent series $\sum |r|^j j^n$. The function $\tau'(x)$ is therefore also bounded.

For the remaining terms of $\Phi(x, r, k)$, in which $j \geqq \frac{1}{2}|x|$, we have if $k > 0$

$$\left|\sum_{j=m+1}^{\infty} \frac{r^j}{(x+j)^k}\right| < \begin{cases} \dfrac{|r|^{\frac{1}{2}|x|}}{|x|^k}(1+|r|+|r|^2+\cdots) & (u > 0), \\ \dfrac{|r|^{\frac{1}{2}|x|}}{|v|^k}(1+|r|+|r|^2+\cdots) & (u < 0). \end{cases}$$

Hence, if we exclude a narrow strip on each side of the negative axis of reals, we may write

$$\sum_{j=m+1}^{\infty} \frac{r^j}{(x+j)^k} = \frac{\tau''(x)}{x^{k+n}},$$

where $\tau''(x)$ is bounded for large values of $x$. The same result is easily seen to hold when $k < 0$. Combining this with our previous results, we have

$$\Phi(x,r,k) = \frac{a_0}{x^k} - k\frac{a_1}{x^{k+1}} + \cdots + \binom{-k}{n}\frac{a_n}{x^{k+n}} + \frac{\tau'(x)+\tau''(x)}{x^{k+n}}.$$

If now we set

$$s_0 = \frac{1}{1-r}, \quad s_1 = -k\frac{r}{(1-r)^2}, \quad s_2 = \binom{-k}{2}\frac{r+r^2}{(1-r)^3}, \text{ etc.,}$$

we have

$$\Phi(x, r, k) = \frac{s_0}{x^k} + \frac{s_1}{x^{k+1}} + \cdots + \frac{s_n}{x^{k+n}} + \frac{\tau(x)}{x^{k+n}},$$

where the function

$$\tau(x) = \tau_0(x) - k\tau_1(x) + \cdots + \binom{-k}{n}\tau_n(x) + \tau'(x) + \tau''(x)$$

is bounded for large values of $x$. Since $n$ is any positive integer, we have by the definition in § 6

$$\Phi(x, r, k) \sim \frac{s_0}{x^k} + \frac{s_1}{x^{k+1}} + \cdots.$$

This holds for the entire plane when $k < 0$, and everywhere except near the negative axis of reals when $k > 0$.

## Exercises.

1. Evaluate $\mathop{\mathsf{S}} x^2$; $\mathop{\mathsf{S}} \frac{1}{x(x+1)}$; $\mathop{\mathsf{S}} \cos x$.

2. Verify that

$$\mathop{\mathsf{S}} u(x)\,\Delta v(x) = u(x)\,v(x) - \mathop{\mathsf{S}} v(x+1)\,\Delta u(x).$$

3. Form the difference equations whose general solutions are

(a) $x p_1(x) + x^2 p_2(x)$;

(b) $x p_1(x) + \frac{1}{x} p_2(x)$;

(c) $x p_1(x) + 2^x p_2(x) + x 2^x p_3(x)$.

(Cf. the method for differential equations.)

4. Prove that the determinant of Casorati satisfies the difference equation

$$D(x+1) = (-1)^n \frac{a_0(x)}{a_n(x)} D(x).$$

5. Find the general solutions of the equations

(a) $2y(x+3) - 5y(x+2) + y(x+1) + 2y(x) = 0$;

(b) $y(x+3) - y(x+2) - 8y(x+1) + 12y(x) = 0$;

(c) $y(x+3) + 6y(x+2) + 12y(x+1) + 8y(x) = 0$;

(d) $y(x+4) - 4y(x+3) + 8y(x+2) - 8y(x+1) + 4y(x) = 0$.

6. Find the general solutions of the equations

(a) $y(x+1) - 2y(x) = 2^x$;

(b) $y(x+2) - 3y(x+1) + 2y(x) = 1$;

(c) $y(x+2) - 2\frac{x+2}{x+1} y(x+1) + \frac{x+2}{x} y(x) = 2$

(cf. ex. 3(a) above).

7. Investigate the asymptotic form of the series

$$\frac{1}{x} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^3} + \cdots$$

(a) in the right half plane; (b) in the left half plane exclusive of the strip $-1 \leq v \leq 1$.

8. Determine the asymptotic form of the integral

$$\int_x^{\infty} \frac{e^t}{t^k}\, dt \qquad (k \neq 0, 1, 2, \cdots),$$

where $x$ lies on the positive axis of reals.

9. Discuss the asymptotic form of the function

$$f(x) = e^x + e^{\omega x} + e^{\omega^2 x},$$

where $\omega$ is a primitive cube root of unity.

10. Determine the asymptotic form of the integral

$$\int_x^{\infty} \frac{e^{x-t}}{t}\, dt$$

throughout the entire plane.

11. Show that $\Phi(x, r, k)$ is identically equal to its asymptotic form when $k < 0$.

12. Investigate the convergence and asymptotic form of $\Phi(x, -1, k)$.

## Chapter II.

# Equations of the first order. The gamma function.

### § 1. Homogeneous equations.

In this section we will study the homogeneous linear equation

$$y(x+1)-r(x)\,y(x)=0, \tag{38}$$

where $r(x)$ is a rational function. If $y_1(x)$ and $y_2(x)$ are any two solutions, their ratio is a periodic function, for

$$\frac{y_1(x+1)}{y_2(x+1)}=\frac{r(x)\,y_1(x)}{r(x)\,y_2(x)}=\frac{y_1(x)}{y_2(x)}.$$

The most general solution of (38) can therefore be obtained by multiplying any particular solution* by an arbitrary periodic function of period 1.

From eq. (38) we have

$$\begin{aligned} y(x) &= r(x-1)\,y(x-1) \\ &= r(x-1)\,r(x-2)\,y(x-2) \\ &\cdots\cdots\cdots\cdots \\ &= r(x-1)\,r(x-2)\cdots r(x-n)\,y(x-n). \end{aligned} \tag{39}$$

Likewise

$$\begin{aligned} y(x) &= \frac{1}{r(x)}\,y(x+1) \\ &= \frac{1}{r(x)}\,\frac{1}{r(x+1)}\,y(x+2) \\ &\cdots\cdots\cdots\cdots \\ &= \frac{1}{r(x)}\,\frac{1}{r(x+1)}\cdots\frac{1}{r(x+n)}\,y(x+n+1). \end{aligned} \tag{40}$$

* In discussing homogeneous difference equations we shall in general disregard the trivial solution $y(x)=0$.

Letting $n$ become infinite, we obtain the two *symbolic solutions*

$$(41)\quad y_l(x) = \prod_{n=1}^{\infty} r(x-n), \quad y_r(x) = \prod_{n=0}^{\infty} \frac{1}{r(x+n)}.$$

To investigate the convergence of these infinite products, write $r(x) = 1 + r'(x)$; the convergence then depends on that of the series $\sum r'(x-n)$ and $\sum r'(x+n)$. Expressing $r'(x)$ as the quotient of two polynomials $a(x)/b(x)$, we see that if the degree of $b(x)$ exceeds that of $a(x)$ by 2 or more, these series converge absolutely and uniformly in any finite region from which the points congruent to the zeros of $b(x)$ have been excluded by small circles drawn about them. In this case the products converge uniformly in any such region, and $y_l(x)$ and $y_r(x)$ are single-valued functions, analytic over the whole plane except for poles. The poles of $y_l(x)$ lie at the points congruent on the right to the poles of $r(x)$, and those of $y_r(x)$ lie at the zeros of $r(x)$ and the points congruent to these on the left.

Let us now attempt to find a power series in $1/x$ which will satisfy eq. (38). The rational function $r(x)$ may be written

$$(42)\qquad r(x) = x^{\mu}\left(c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots\right),$$

where $\mu$ is an integer and $c_0 \neq 0$. Direct substitution shows that a simple series cannot in general satisfy eq. (38), so we will multiply it by some suitable exponential factors, and substitute

$$y(x) = x^{ax}\, b^x\, x^d \left(s_0 + \frac{s_1}{x} + \frac{s_2}{x^2} + \cdots\right)$$

in eq. (38). Using the expansion

$$\begin{aligned}
(x+1)^{a(x+1)} &= (x+1)^a x^{ax}\left(1+\frac{1}{x}\right)^{ax} \\
&= (x+1)^a x^{ax} e^{ax\log\left(1+\frac{1}{x}\right)} \\
&= (x+1)^a x^{ax} e^{ax\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}-\cdots\right)} \\
(43)\qquad &= x^{ax+a}\left(1+\frac{1}{x}\right)^a e^a\left[1-\frac{a}{2x}+\frac{1}{x^2}\left(\frac{a^2}{8}+\frac{a}{3}\right)-\cdots\right]
\end{aligned}$$

and removing the factor $x^{ax}\, b^x\, x^d$ from both sides, we have

$$b\,x^a\, e^a\left(1-\frac{a}{2x}+\cdots\right)\left(1+\frac{1}{x}\right)^{a+d}\left[s_0+\frac{s_1}{x}\left(1+\frac{1}{x}\right)^{-1}+\cdots\right]$$
$$= x^{\mu}\left(c_0+\frac{c_1}{x}+\cdots\right)\left(s_0+\frac{s_1}{x}+\cdots\right).$$

Expanding both sides into power series in $1/x$, and equating the coefficients of like powers, we see that we must have

$$a=\mu, \qquad b\,e^a\, s_0 = c_0\, s_0,$$
$$b\,e^a\left(\frac{a\,s_0}{2}+d\,s_0+s_1\right) = c_1\, s_0 + c_0\, s_1, \text{ etc.}$$

These equations serve to determine the quantities $a$, $b$, $d$, $\frac{s_1}{s_0}, \frac{s_2}{s_0}, \cdots$, namely

$$a=\mu, \qquad b = c_0\, e^{-\mu},$$
$$d = \frac{c_1}{c_0}-\frac{\mu}{2}, \qquad \frac{s_1}{s_0} = \frac{c_1}{2\,c_0}\left(\frac{c_1}{c_0}-1\right)-\frac{c_2}{c_0}+\frac{\mu}{12}, \text{ etc.};$$

$s_0$ is arbitrary, since a solution may obviously be multiplied by an arbitrary constant. The equation from which $s_k/s_0$ is determined is

$$b\,e^a\left[s_{k+1}-k\,s_k+s_k\left(-\frac{a}{2}+a+d\right)+\cdots\right]$$
$$= c_0\, s_{k+1}+c_1\, s_k+\cdots,$$

and if we put in the values already obtained for $a$, $b$, and $d$ we see that the terms in $s_{k+1}$ cancel and the coefficient of $s_k$ reduces to $-k\,c_0$, which is different from zero. Hence all the coefficients of the series can be uniquely determined, apart from the arbitrary constant multiplier. Eq. (38) is therefore satisfied formally by the series

$$(41) \quad S(x) = x^{\mu x}\, e^{-\mu x}\, c_0^x\, x^{\frac{c_1}{c_0}-\frac{\mu}{2}}\left(s_0+\frac{s_1}{x}+\frac{s_2}{x^2}+\cdots\right).$$

We naturally inquire under what conditions this series converges. If it does converge for $|x| > R$, $S(x)$ represents

an analytic solution of eq. (38) in this region. If $x$ makes a negative circuit about the point $\infty$, $S(x)$ is multiplied by $e^{2\mu\pi i x}\, e^{2\pi i\left(\frac{c_1}{c_0}-\frac{\mu}{2}\right)}$, on account of the multiple-valued factors $x^{\mu x}$ and $x^{\frac{c_1}{c_0}-\frac{\mu}{2}}$ But an analytic solution of eq. (38) cannot be multiple-valued, as we shall see when we obtain the general solution [cf. eqs. (51) and (74)]; hence we must have $\mu = 0$ and $c_1/c_0 =$ an integer. If this condition is satisfied, the function

$$S(x)\, c_0^{-x} = x^{\frac{c_1}{c_0}}\left(s_0 + \frac{s_1}{x} + \frac{s_2}{x^2} + \cdots\right)$$

is analytic outside the circle $|x| = R$, except for a possible pole at $x = \infty$; inside this circle it has no singularities except poles, as is evident from eqs. (39) and (40), in which $n$ may be taken so large that $x-n$ and $x+n+1$ lie outside the circle; hence it must be a rational function $R(x)$. Substituting $c_0^x\, R(x)$ for $y(x)$ in eq. (38), we see that

$$r(x) = c_0 \frac{R(x+1)}{R(x)}. \tag{45}$$

Conversely, if $r(x)$ can be written in this form, eq. (38) has the solution $y(x) = c_0^x\, R(x)$, in which $R(x)$ may be expanded into a convergent power series in $\frac{1}{x}$. Hence eq. (45) is a necessary and sufficient condition for the convergence of $S(x)$.

We see, then, that eq. (38) is satisfied formally by two symbolic solutions $y_l(x)$ and $y_r(x)$ and by a power series $S(x)$, which may converge under certain conditions, but in general do not. Solutions of these types are characteristic of the most general homogeneous linear difference equation, and the essence of the proofs of our existence theorems will be a suitable modification of the symbolic solutions, with the aid of the series, so that they will converge. We proceed to obtain by this method two analytic solutions of eq. (38).

Let $T(x)$ denote the sum of the first $k$ terms of $S(x)$, i. e., $T(x) = a(x)\,S_k(x)$, where $a(x)$ denotes the product of the exponential factors and $S_k(x)$ the sum of the first $k$ terms of the power series in (44). For large values of $x$, $T(x)$ may be regarded as an *approximate solution* of eq. (38); for we may write

$$S(x) = a(x)\left[S_k(x) + \frac{s_k}{x^k} + \frac{\sigma(x)}{x^{k+1}}\right],$$

whence we have formally

$$\frac{T(x+1)}{T(x)} \div \frac{S(x+1)}{S(x)} = \frac{S_k(x+1)\left[S_k(x) + \dfrac{s_k}{x^k} + \dfrac{\sigma(x)}{x^{k+1}}\right]}{S_k(x)\left[S_k(x+1) + \dfrac{s_k}{(x+1)^k} + \dfrac{\sigma(x+1)}{(x+1)^{k+1}}\right]};$$

dividing numerator and denominator by $S_k(x)\,S_k(x+1)$, and expanding in powers of $\frac{1}{x}$, we see that this is equal to

$$\frac{1 + \dfrac{s_k}{s_0\,x^k} + \dfrac{\sigma'(x)}{x^{k+1}}}{1 + \dfrac{s_k}{s_0\,x^k} + \dfrac{\sigma''(x)}{x^{k+1}}} = 1 + \frac{\theta(x)}{x^{k+1}},$$

where $\sigma'(x)$, $\sigma''(x)$, and $\theta(x)$ are all power series in $\frac{1}{x}$, and $\theta(x)$ is convergent for large values of $x$, since $\frac{S(x+1)}{S(x)}$ is formally equal to $r(x)$, i. e., to the convergent series (42). Hence $\frac{T(x+1)}{T(x)}$ differs from $\frac{S(x+1)}{S(x)}$ or $r(x)$ only by terms of the $(k+1)$st degree, so we have

$$\text{(46)} \qquad \frac{T(x+1)}{T(x)} = r(x)\left[1 + \frac{\theta(x)}{x^{k+1}}\right],$$

where $|\theta(x)| < M$ for $|x| > R$. It is in this sense that $T(x)$ is an approximate solution of equation (38) for large values of $x$.

In eqs. (39) and (40) replace $y(x-n)$ and $y(x+n+1)$ by $T(x-n)$ and $T(x+n+1)$ respectively; if $n$ is large in comparison with $x$, we obtain thus two analytic approximate solutions. Since the approximation appears to get closer as $n$ increases, we naturally let $n \to \infty$ and investigate the expressions

$$(47)\quad \begin{cases} h(x) = \lim\limits_{n\to\infty} \dfrac{1}{r(x)} \dfrac{1}{r(x+1)} \cdots \dfrac{1}{r(x+n)} T(x+n+1), \\ g(x) = \lim\limits_{n\to\infty} r(x-1)\, r(x-2) \cdots r(x-n)\, T(x-n). \end{cases}$$

If these limits exist, $h(x)$ and $g(x)$ are solutions of eq. (38), as may be readily verified by substitution. Moreover, the solutions are independent of the value of $k$; for if $T'(x)$ denotes the sum of the first $k'$ terms of $S(x)$, then for any fixed value of $x$

$$\lim_{n\to\infty} \frac{T'(x+n+1)}{T(x+n+1)} = \lim_{x\to\infty} \frac{T'(x-n)}{T(x-n)} = 1.$$

To prove the convergence of $h(x)$, let

$$(48)\quad H_n(x) = \frac{1}{r(x)} \frac{1}{r(x+1)} \cdots \frac{1}{r(x+n)} T(x+n+1);$$

this may be written

$$H_n(x) = T(x) \frac{T(x+1)}{r(x)\, T(x)} \frac{T(x+2)}{r(x+1)\, T(x+1)} \cdots \frac{T(x+n+1)}{r(x+n)\, T(x+n)},$$

or, by (46),

$$H_n(x) = T(x) \left[1 + \frac{\theta(x)}{x^{k+1}}\right] \left[1 + \frac{\theta(x+1)}{(x+1)^{k+1}}\right] \cdots \left[1 + \frac{\theta(x+n)}{(x+n)^{k+1}}\right].$$

Accordingly we have

$$(49)\quad h(x) = \lim_{n\to\infty} H_n(x) = T(x) \prod_{n=0}^{\infty} \left[1 + \frac{\theta(x+n)}{(x+n)^{k+1}}\right].$$

If $x$ is in the right half plane and $|x| > R$, or if $x$ is in the left half plane and $|v| > R$, then $|\theta(x+n)| < M$ for every value

of $n$. The infinite product converges uniformly in the neighborhood of any such point, since the series $\sum \frac{\theta(x+n)}{(x+n)^{k+1}}$ then converges absolutely and uniformly. Moreover the relation

$$H_n(x) = \frac{1}{r(x)} \frac{1}{r(x+1)} \cdots \frac{1}{r(x+m-1)} H_{n-m}(x+m),$$

which follows directly from (48), shows that $H_n(x)$ converges also in the region excluded above; for $m$ may be taken so large that $x+m$ lies in the right half plane and $|x+m| > R$. The only points at which $H_n(x)$ does not converge are the zeros of $r(x)$ and the points congruent to them on the left.

The function $h(x)$ defined by eq. (47) is therefore a solution of eq. (38) which is analytic throughout the finite part of the plane except at certain isolated sets of points. Since a convergent infinite product can vanish only when one of its factors vanishes, eq. (47) enables us to locate precisely all the poles and zeros of $h(x)$. The poles lie at the zeros of $r(x)$ and points congruent to them on the left, and the zeros lie at the poles of $r(x)$ and points congruent on the left. The orders of the poles and zeros are of course the same as those of the corresponding zeros and poles of $r(x)$. In case $r(x)$ has a zero of order $\lambda$ and a pole of order $\mu$ at two congruent points, then at points congruents on the left to both $h(x)$ will have a pole of order $\lambda - \mu$ or a zero of order $\mu - \lambda$, according as $\lambda > \mu$ or $\lambda < \mu$; if $\lambda = \mu$, $h(x)$ will be analytic and different from zero at such points.

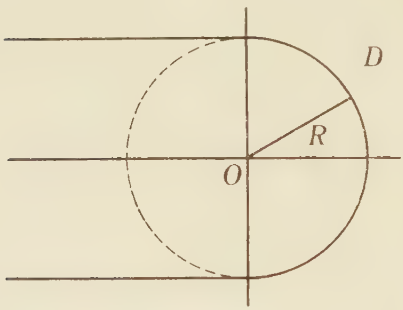

Fig. 1.

At the point $\infty$ $h(x)$ in general has an essential singularity. On account of this distribution of poles and zeros, it is evident that if a circle

of sufficiently large radius $R$ is drawn about the origin to include all the finite poles and zeros of $r(x)$, and lines tangent to this circle are drawn parallel to the negative axis of reals, all the singularities of $h(x)$ will lie in the region thus enclosed, and in the region $D$ outside it will be analytic and different from zero.

It remains to discuss the behavior of $h(x)$ in the neighborhood of $x = \infty$. If $x$ lies in the region $D$, for which $R$ is taken sufficiently large, we have from eq. (49)

$$\left|\frac{h(x)}{T(x)} - 1\right| = \left|\prod_{n=0}^{\infty}\left[1 + \frac{\theta(x+n)}{(x+n)^{k+1}}\right] - 1\right|$$

$$\leqq \prod_{n=0}^{\infty}\left[1 + \frac{M}{|x+n|^{k+1}}\right] - 1$$

$$\leqq \prod_{n=0}^{\infty} e^{\frac{M}{|x+n|^{k+1}}} - 1 = e^{M\sum_{n=0}^{\infty}\frac{1}{|x+n|^{k+1}}} - 1;$$

hence by the inequalities (28) and (29)

$$\left|\frac{h(x)}{T(x)} - 1\right| < \begin{cases} e^{\frac{M}{|x|^k}\left(\frac{1}{|x|} + \frac{1}{2}\right)} - 1 & (u \geqq 0), \\ e^{\frac{2M}{|v|^k}\left(\frac{1}{|v|} + \frac{\pi}{2}\right)} - 1 & (u < 0). \end{cases}$$

The right hand members vanish at $x = \infty$ to the $k$th order in $x$ and $v$ respectively. Hence if $x \to \infty$ in any direction except parallel to the negative axis of reals,

$$\lim_{x \to \infty} x^{k-1}\left[\frac{h(x)}{T(x)} - 1\right] = 0;$$

this is true in the left half plane as well as the right, since along any ray through the origin $|v|$ is proportional to $|x|$. Using the definition of $T(x)$, we have

$$\lim_{x \to \infty} x^{k-1}\left[\frac{h(x)}{x^{\mu x}(c_0 e^{-\mu})^x x^{\frac{c_1}{c_0} - \frac{\mu}{2}}} - \left(s_0 + \frac{s_1}{x} + \cdots + \frac{s_{k-1}}{x^{k-1}}\right)\right] = 0;$$

since this is true for every $k$, $h(x) \sim S(x)$ in the sector $-\pi < \arg x < \pi$.

A similar discussion may be made of the solution $g(x)$ defined by (47). The results for both solutions may be stated as follows.

THEOREM. *The linear homogeneous difference equation*

$$y(x+1) - r(x)\, y(x) = 0$$

*is satisfied formally by the series*

$$S(x) = x^{\mu x} c_0^x e^{-\mu x} x^{\frac{c_1}{c_0} - \frac{\mu}{2}} \left( s_0 + \frac{s_1}{x} + \cdots \right);$$

*there exist also two analytic solutions*

$$h(x) = \lim_{n \to \infty} \frac{1}{r(x)} \frac{1}{r(x+1)} \cdots \frac{1}{r(x+n)} T(x+n+1),$$

$$g(x) = \lim_{n \to \infty} r(x-1)\, r(x-2) \cdots r(x-n)\, T(x-n),$$

*where $T(x)$ is the sum of the first $k$ terms of $S(x)$.*

*The solution $h(x)$ is analytic throughout the plane except for poles at the zeros of $r(x)$ and points congruent on the left; it vanishes at the poles of $r(x)$ and points congruent on the left; it is represented asymptotically by $S(x)$ in the sector $-\pi < \arg x < \pi$. The solution $g(x)$ is analytic throughout the plane except for poles at points congruent on the right to the poles of $r(x)$; it vanishes at points congruent on the right to the zeros of $r(x)$; it is represented asymptotically by $S(x)$ in the sector $0 < \arg x < 2\pi$.*

The solutions $h(x)$ and $g(x)$ are called the *first principal solution* and the *second principal solution* respectively. They are uniquely determined (apart from a constant factor) by their asymptotic properties. For suppose a solution $h'(x)$ different from $h(x)$ were also represented asymptotically by $S(x)$ in the right half plane; consider their ratio, which, as we have seen, is a periodic function $p(x)$. Let $x \to \infty$ along a line parallel to the axis of reals; since $h(x)$ and $h'(x)$ have the same asymptotic form, their ratio $p(x) \to 1$; but since

$p(x)$ is periodic, it must therefore be equal to 1, i. e., $h'(x) = h(x)$. This argument may be applied if only the first term of the asymptotic form is known, for even then the ratio approaches 1. Similarly, we can show that $g(x)$ is the only solution which is represented asymptotically by $S(x)$ in the left half plane.

## § 2. The gamma function.

Among the equations of the form (38) there is one of particular importance, namely

$$y(x+1) = xy(x). \tag{50}$$

If $G(x)$ denotes any solution of this, we can solve eq. (38) in terms of $G(x)$; for if we write $r(x)$ in the factored form

$$r(x) = c_0 \frac{(x-\alpha_1)\ (x-\alpha_2)\ \cdots\ (x-\alpha_m)}{(x-\beta_1)\ (x-\beta_2)\ \cdots\ (x-\beta_n)}\,,$$

it is easily seen that (38) is satisfied by

$$y(x) = c_0^x \frac{G(x-\alpha_1)\ G(x-\alpha_2)\ \cdots\ G(x-\alpha_m)}{G(x-\beta_1)\ G(x-\beta_2)\ \cdots\ G(x-\beta_n)}\,; \tag{51}$$

the general solution of (38) is obtained by multiplying this by an arbitrary periodic function.

The first principal solution of eq. (50) is called the *gamma function,* and is denoted by $\Gamma(x)$. This function, which possesses many interesting and elegant properties, was first introduced into analysis by Euler, about 1730. Since that time it has been extensively studied.* It has been customary to define the gamma function either as an infinite product or as a definite integral [eqs. (53), (69)], and to deduce incidentally the fact that it satisfies the difference equation (50). Here, however, we make the difference equation fundamental, and we shall see that the most important properties of the function are readily obtained from this standpoint.

* See Nielsen: *Handbuch der Theorie der Gammafunktion.* (Leipzig, 1906.)

The formal series for eq. (50) is

$$(52)\quad S(x) = x^x e^{-x} x^{-\frac{1}{2}} s_0 \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \cdots\right),$$

since here $\mu = 1$, $c_0 = 1$, $c_1 = c_2 = \cdots = 0$; $s_0$ is a constant, which for the moment we will leave arbitrary.

If we take for $T(x)$ the first term of $S(x)$, we have from (47), using $n-1$ instead of $n$,

$$\Gamma(x) = \lim_{n\to\infty} \frac{1}{x}\,\frac{1}{x+1} \cdots \frac{1}{x+n-1}\,(x+n)^{x+n-\frac{1}{2}} e^{-x-n} s_0:$$

similarly,

$$\Gamma(1) = \lim_{n\to\infty} \frac{1}{1}\cdot\frac{1}{2} \cdots \frac{1}{n}\,(n+1)^{n+\frac{1}{2}} e^{-n-1} s_0,$$

and by division we have

$$\frac{\Gamma(x)}{\Gamma(1)} = \lim_{n\to\infty} \frac{n!\; n^{x-1}}{x(x+1)\cdots(x+n-1)} \left(\frac{x+n}{n}\right)^{x-1} \left(\frac{x+n}{n+1}\right)^{n+\frac{1}{2}} e^{1-x}.$$

The last two factors cancel each other, since

$$\lim_{n\to\infty} \left(\frac{x+n}{n+1}\right)^{n+\frac{1}{2}} = \lim_{n\to\infty} \left(1 + \frac{x-1}{n+1}\right)^{n+\frac{1}{2}} = e^{x-1}.$$

For the preceding factor we have

$$\lim_{n\to\infty} \left(\frac{x+n}{n}\right)^{x-1} = \lim_{n\to\infty} e^{(x-1)\log\left(1+\frac{x}{n}\right)} = e^{2l\pi i(x-1)},$$

where $l$ may be any integer. In order that $\Gamma(x)$ shall have a definite meaning, it is necessary to choose a particular determination of the multiple-valued factor $x^{x-\frac{1}{2}} = e^{\left(x-\frac{1}{2}\right)\log x}$ in $S(x)$; let us take $-\pi < \arg x \leq \pi$, so that $x^{x-\frac{1}{2}}$ is real when $x$ is real and positive; then the arguments of $x+n$ and $1+x/n$ approach $0$ as $n\to\infty$, whence $l=0$ and the

limit above is 1. Let us now choose the constant $s_0$ so that $\Gamma(1) = 1$.* Then

$$(53) \qquad \Gamma(x) = \lim_{n\to\infty} \frac{(n-1)!\, n^x}{x(x+1)\cdots(x+n-1)}.$$

By the theorem of § 1, $\Gamma(x)$ is analytic throughout the finite plane except for poles of the first order at $x = 0, -1, -2, \cdots$; it has no zeros. For large values of $x$, $\Gamma(x)$ is represented asymptotically by the series (52) in the sector $-\pi < \arg x < \pi$. Since $\Gamma(1) = 1$, we have directly from eq. (50)

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1,\ \Gamma(3) = 2\cdot 1, \cdots, \Gamma(n+1) = n!,$$

where $n$ is any positive integer. By eq. (53), the residue of $\Gamma(x)$ at the pole $x = -k$ is

$$\lim_{x\to -k} (x+k)\,\Gamma(x) = \lim_{n\to\infty} \frac{(-1)^k}{k!} \frac{(n-k)(n-k+1)\cdots(n-1)}{n^k} = \frac{(-1)^k}{k!}.$$

The infinite product (53) may be written also in the following forms:

$$(54) \quad \Gamma(x) = \frac{1}{x}\prod_{n=1}^{\infty} \frac{(n+1)^x}{n^{x-1}(x+n)} = \frac{1}{x}\prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^x}{1+\frac{x}{n}}.$$

Another standard expression for $\Gamma(x)$ may be obtained as follows:

$$\Gamma(x) = \lim_{n\to\infty} e^{x\log n}\,\frac{1}{x}\,\frac{1}{x+1}\,\frac{2}{x+2}\cdots\frac{n-1}{x+n-1}$$

$$= \lim_{n\to\infty} e^{-x\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\log n\right)}\, e^{x+\frac{x}{2}+\cdots+\frac{x}{n-1}}\,\frac{1}{x}\cdots\frac{n-1}{x+n-1}$$

$$(55) \quad = \frac{e^{-Cx}}{x}\prod_{n=1}^{\infty} e^{\frac{x}{n}}\,\frac{n}{x+n},$$

* The required value is $\sqrt{2\pi}$, as we shall see later (p. 47).

where

$$(56) \quad C = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}-\log n\right).$$

To prove that this limit exists, we note that (56) may be written

$$C = \lim_{n\to\infty}\sum_{s=1}^{n-1}\left(\frac{1}{s}-\log\frac{s+1}{s}\right) = \sum_{s=1}^{\infty}\frac{1}{2\left(1+\frac{\theta_s}{s}\right)^2 s^2},$$

where $0<\theta_s<1$, for by Taylor's Theorem

$$\log\frac{s+1}{s} = \log\left(1+\frac{1}{s}\right) = \frac{1}{s}-\frac{1}{\left(1+\frac{\theta_s}{s}\right)^2}\frac{1}{2s^2}.$$

Since $\theta_s$ and $s$ are positive, the series is term by term less than the convergent series $\sum\frac{1}{2s^2}$; therefore the limit $C$ exists. It is known as *Euler's constant*, and has the numerical value

$$C = 0.5772156649\cdots.$$

The second principal solution of eq. (50) we will denote by $\bar{\Gamma}(x)$. By the theorem of § 1 it is analytic throughout the finite plane, and has zeros of the first order at $x = 1, 2, 3, \cdots$; since it has no poles, it is an entire function. It is represented asymptotically by the series (52) in the sector $0 \leqq \arg x < 2\pi$.

The function $\bar{\Gamma}(x)$, like $\Gamma(x)$, is not completely determined until we have chosen a value for the constant $s_0$ and a particular determination of $x^{x-\frac{1}{2}}$. We will give $s_0$ the same value as we did for $\Gamma(x)$, and take $0 < \arg x < 2\pi$. It follows that $\Gamma(x)$ and $\bar{\Gamma}(x)$ are represented asymptotically by the same determination of $S(x)$ in the upper half plane, but in the lower half plane the determination for $\bar{\Gamma}(x)$ is obtained from that for $\Gamma(x)$ by increasing the argument of $x$ by $2\pi$, which changes $S(x)$ to $e^{2\pi i\left(x-\frac{1}{2}\right)}S(x)$.

The function $\bar{\Gamma}(x)$ can be expressed in terms of $\Gamma(x)$. Since they are both solutions of eq. (50), their ratio must

be a periodic function; let $p(x) = \Gamma(x)/\Gamma(x)$, and consider the behavior of $p(x)$ in any period strip, say $0 \leq u < 1$. From the properties of $\Gamma(x)$ and $\bar{\Gamma}(x)$ it is evident that $p(x)$ has no poles, and that is vanishes only at $x = 0$, so it is analytic throughout the finite part of the strip. At the upper end of the strip $p(x) \to 1$, since $\Gamma(x)$ and $\Gamma(x)$ are represented asymptotically there by the same determination of $S(x)$. At the lower end of the strip $\Gamma(x) \sim S(x)$ and $\Gamma(x) \sim e^{2\pi i\left(x \frac{1}{2}\right)} S(x)$, so $p(x)$ behaves like $e^{\cdot 2\pi i\left(x - \frac{1}{2}\right)} = -e^{2\pi i x}$, which becomes infinite as $x \to \infty$ in the strip. Hence, by the theorem of § 2, Chap. I, $p(x)$ is a rational function of $e^{2\pi i x}$. Since $p(x)$ vanishes to the first order at $x = 0$, it must contain a factor $e^{2\pi i x} - 1$ in the numerator. Since it has no finite poles, the denominator can contain no factor of the form $e^{2\pi i x} - a \, (a \neq 0)$, and hence $p(x)$ must have the form $c e^{2k\pi i x}(e^{2\pi i x} - 1)$. The behavior of $p(x)$ at the lower end of the strip shows that $k = 0$, $c = -1$; hence we have

$$\Gamma(x) = (1 - e^{2\pi i x})\, \Gamma(x). \tag{57}$$

This formula enables us to study the behavior of $\Gamma(x)$ as $x \to \infty$ along a ray parallel to the negative axis of reals. Since $\Gamma(x) \sim S(x)$ along such a ray, we have

$$\Gamma(x) \sim \frac{1}{1 - e^{2\pi i x}} S(x).$$

Suppose first that the ray is above the axis of reals; the periodic function differs from 1 by a quantity of the order of $e^{-2\pi v}$, so we can write

$$\Gamma(x) \sim [1 + \pi(x)]\, S(x),$$

where $\pi(x)$ is a periodic function which approaches zero exponentially as $v$ increases. Hence $\Gamma(x) \sim S(x)$ to a degree of approximation which can be made arbitrarily close by taking the ray far enough above the axis of reals.

If the ray is below the axis of reals, $v$ is negative, and the periodic function differs from $-e^{-2\pi i x}$ by a quantity of

the order of $e^{-2\pi|v|}$. Hence $\Gamma(x) \sim -e^{-2\pi i x} S(x)$ to an arbitrary degree of approximation if the ray is sufficiently far below the axis of reals. Here $S(x)$ denotes the determination of the series for $\overline{\Gamma}(x)$, i. e., for $\arg x$ near $\pi$, which, as we saw above, is equal to that for $\arg x$ near $-\pi$ multiplied by $-e^{2\pi i x}$. If we use the latter determination, as is more natural here, we have $\Gamma(x) \sim S(x)$.

In the same way we see that $\Gamma(x) \sim S(x)$ to an arbitrary degree of approximation along rays parallel to the positive axis of reals and sufficiently far above or below it.

The gamma function is connected with the trigonometric functions by the relation

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \tag{58}$$

which was discovered by Euler. To prove this, write $\Gamma(1-x) = -x\Gamma(-x)$, by eq. (50), and express $\Gamma(x)$ and $\Gamma(-x)$ by means of the second product in (54); we thus find that

$$\Gamma(x)\Gamma(1-x) = \frac{1}{x}\prod_{n=1}^{\infty}\frac{1}{\left(1-\frac{x}{n}\right)\left(1-\frac{x}{n}\right)}$$

$$= \frac{1}{x}\prod_{n=1}^{\infty}\frac{1}{\left(1-\frac{x^2}{n^2}\right)}.$$

A comparison of this with the infinite product for $\sin x$, namely

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

at once gives us (58). For $x = \frac{1}{2}$, eq. (58) yields the numerical result $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

By expressing $\sin \pi x$ in terms of $e^{\pi i x}$, we can write (58) in the form

$$\Gamma(x)\Gamma(1-x) = \frac{-2\pi i e^{\pi i x}}{1-e^{2\pi i x}}. \tag{59}$$

this and (57) give us another expression for $\Gamma(x)$:

$$(60) \qquad \Gamma(x) = \frac{-2\pi i e^{\pi i x}}{\Gamma(1-x)}.$$

We will now determine the value of the constant $s_0$ in (52). Since $\Gamma(x) \sim S(x)$, we may write for large values of $x$ (except near the negative axis of reals)

$$(61) \qquad \Gamma(x) = x^{x-\frac{1}{2}} e^{-x} s_0 [1 + \varepsilon(x)],$$

where $\lim\limits_{x \to \infty} \varepsilon(x) = 0$, and $x^{x-\frac{1}{2}}$ denotes the branch of the function for $-\pi < \arg x < \pi$. Hence

$$(62) \qquad \begin{aligned} \Gamma(x)\,\Gamma(1-x) &= -x\,\Gamma(x)\,\Gamma(-x) \\ &= (-1)^{-x+\frac{1}{2}} s_0^2 [1 + \varepsilon'(x)], \end{aligned}$$

where $\lim\limits_{x \to \infty} \varepsilon'(x) = 0$. Since the arguments of both $x$ and $-x$ must lie between $-\pi$ and $\pi$, if we take $x$ in the upper half plane we have $\arg(-x) = \arg x - \pi$, so that the factor $-1$, which represents the ratio of $-x$ to $x$, has the argument $-\pi$. Hence in the upper half plane

$$(-1)^{-x+\frac{1}{2}} = (e^{-\pi i})^{-x+\frac{1}{2}} = -i e^{\pi i x}.$$

Equating the values of $\Gamma(x)\,\Gamma(1-x)$ in (62) and (59), we have for large values of $x$ not near the axis of reals

$$s_0^2 [1 + \varepsilon'(x)] = \frac{2\pi}{1 - e^{2\pi i x}}.$$

If now we let $x \to \infty$ parallel to the positive axis of imaginaries, we get $s_0^2 = 2\pi$, or $s_0 = \sqrt{2\pi}$ (the positive root is taken, since $\Gamma(x)$ is positive when $x$ is real and positive). Putting this value in (61), we have

$$(63) \quad \Gamma(x) = x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi}\, [1 + \varepsilon(x)], \qquad \lim_{x \to \infty} \varepsilon(x) = 0,$$

in the sector $-\pi < \arg x < \pi$, or, by (52),

$$\Gamma(x) \sim x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \cdots\right). \tag{64}$$

By division (cf. the expansion (43)) we obtain the following two results, which we shall need later:

$$\left\{\begin{aligned} \frac{\Gamma(x+c)}{\Gamma(x)} &\sim \frac{S(x+c)}{S(x)} = x^c\left(1 + \frac{c^2-c}{2x} + \cdots\right), \\ \frac{\Gamma(x)}{\Gamma(x-c)} &\sim \frac{S(x)}{S(x-c)} = x^c\left(1 - \frac{c^2+c}{2x} + \cdots\right), \end{aligned}\right. \tag{65}$$

where $c$ is any constant. These hold in the same sector $-\pi < \arg x < \pi$.

The gamma function satisfies the relation

$$\Gamma(nx) = \frac{n^{nx-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}} \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right), \tag{66}$$

which is known as *Gauss's multiplication theorem*; it is useful in the numerical computation of $\Gamma(x)$. To prove this, let $\varphi(x)$ denote the product in (66); then

$$\varphi\left(x + \frac{1}{n}\right) = \prod_{k=1}^{n} \Gamma\left(x + \frac{k}{n}\right) = x\varphi(x). \tag{67}$$

This difference equation with interval $1/n$ may be transformed into one with interval 1 by setting $x = z/n$, $\varphi(z/n) = f(z)$, which gives

$$f(z+1) = \frac{z}{n} f(z);$$

a solution of this is by (51) $f(z) = n^{-z}\Gamma(z)$, whence a solution of eq. (67) is $\varphi'(x) = n^{-nx}\Gamma(nx)$. Since $\varphi(x)$ and $\varphi'(x)$ are both solutions of eq. (67), their ratio $\varphi(x)/\varphi'(x) = p(x)$ must

be a periodic function of period $1/n$. Consider the behavior of $p(x)$ in some period strip, say $1 \leq u < 1 + 1/n$; since neither $\varphi(x)$ nor $\varphi'(x)$ vanishes or has any poles in the strip, $p(x)$ is analytic and different from zero in the entire finite part of the strip. To investigate $p(x)$ at the ends of the strip, we use the asymptotic forms of $\varphi(x)$ and $\varphi'(x)$:

$$p(x) = \frac{\prod_{k=0}^{n-1} \left(x + \frac{k}{n}\right)^{x + \frac{k}{n} - \frac{1}{2}} \cdot e^{-nx - \sum_{k=1}^{n-1} \frac{k}{n}} (2\pi)^{\frac{n}{2}}}{n^{-nx} (nx)^{nx - \frac{1}{2}} e^{-nx} \sqrt{2\pi}} [1 + \eta(x)],$$

where $\lim_{x \to \infty} \eta(x) = 0$; this holds at both ends of the strip. Since

$$\lim_{x \to \infty} \frac{\left(x + \frac{k}{n}\right)^{x + \frac{k}{n} - \frac{1}{2}} e^{-\frac{k}{n}}}{x^{x + \frac{k}{n} - \frac{1}{2}}}$$

$$= \lim_{x \to \infty} \left(1 + \frac{k}{nx}\right)^{x} e^{-\frac{k}{n}} \left(1 + \frac{k}{nx}\right)^{\frac{k}{n} - \frac{1}{2}} = 1,$$

we may write

$$p(x) = \frac{\prod_{k=0}^{n-1} x^{x + \frac{k}{n} - \frac{1}{2}} (2\pi)^{\frac{n-1}{2}}}{n^{-\frac{1}{2}} x^{nx - \frac{1}{2}}} [1 + \eta'(x)]$$

$$= n^{\frac{1}{2}} (2\pi)^{\frac{n-1}{2}} [1 + \eta'(x)].$$

Thus $p(x)$ is a function analytic throughout the entire strip and finite at $\infty$; it must therefore be a constant, namely $n^{\frac{1}{2}} (2\pi)^{\frac{n-1}{2}}$. Hence

$$\varphi(x) = n^{\frac{1}{2}} (2\pi)^{\frac{n-1}{2}} \varphi'(x),$$

from which (66) at once follows.

In addition to the various infinite product expressions which we have derived, the gamma function can be expressed

as a definite integral of a type which plays an important part in the theory of linear difference equations (cf. § 7, Chap. III). To prove this, let us write

$$G(x) = \int_a^b t^{x-1}\, v(t)\, dt, \tag{68}$$

and investigate whether it is possible to determine the unknown function $v(t)$ and the path of integration so that $G(x)$ shall be a solution of eq. (50). Substituting this integral for $y(x)$ in eq. (50), we obtain the condition

$$\int_a^b t^x\, v(t)\, dt - \int_a^b x\, t^{x-1}\, v(t)\, dt = 0,$$

or, if we integrate the second term by parts,

$$\int_a^b t^x\, v(t)\, dt - [t^x\, v(t)]_a^b + \int_a^b t^x\, v'(t)\, dt = 0.$$

This condition is satisfied if $v'(t) = -v(t)$, whence $v(t) = ce^{-t}$, provided the path of integration is so chosen that $t^x\, v(t)$ has the same value at both ends. If $u$, the real part of $x$, is positive, we can take the path from 0 to $\infty$ along any ray in the right half plane, and in particular along the positive axis of reals. Thus

$$G(x) = c\int_0^\infty t^{x-1}\, e^{-t}\, dt$$

is a solution of eq. (50). This is an improper integral, which converges uniformly in the neighborhood of every point in the right half plane, and so represents there a single-valued analytic function without singularities. Let us impose the condition that $G(1) = 1$; an evaluation of the integral shows that we must have $c = 1$.

In order to identify $G(x)$ with $\Gamma(x)$, we proceed to study the periodic function $p(x) = G(x)/\Gamma(x)$.* Set $t = x\tau$; then

$$G(x) = x^x \int_0^\infty \tau^{x-1}\, e^{-x\tau}\, d\tau,$$

* Cf. Birkhoff: *Bull. Amer. Math. Soc.*, (2), 20 (1913), pp. 7–8.

where the path of integration may be taken as the axis of reals ($\arg \tau = 0$). This can be expressed as the sum of two integrals with the limits 0,1 and $1,\infty$. Let $x$ be any point in the period strip $1 \leqq u < 2$; then

$$\left| \int_0^1 \tau^{x-1} e^{-x\tau}\, d\tau \right| \leqq \int_0^1 \tau^{u-1} e^{-u\tau}\, d\tau$$

$$\leqq \int_0^1 e^{-\tau}\, d\tau = 1 - \frac{1}{e};$$

$$\left| \int_1^\infty \tau^{x-1} e^{-x\tau}\, d\tau \right| \leqq \int_1^\infty \tau^{u-1} e^{-u\tau}\, d\tau$$

$$< \int_0^\infty \tau e^{-\tau}\, d\tau = G(2) - 1,$$

since by eq. (50) $G(2) = G(1)$. Hence in this period strip $|G(x)| < 2\,|x^x|$. For large values of $x$ in the strip we have

$$|\Gamma(x)| = \left| x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi}\,(1+\varepsilon) \right| > |x^{x-1}|.$$

These inequalities show that $|p(x)| < 2\,|x|$ at both ends of the period strip. Write $p(x) = q(z)$, where $z = e^{2\pi i x}$; this transformation maps the unit strip in the $x$-plane in a (1, 1) manner on the entire $z$-plane. Since both $G(x)$ and $\Gamma(x)$ are single-valued and analytic in the finite part of the strip, and since $\Gamma(x) \neq 0$, $q(z)$ is single-valued and analytic at every point of the $z$-plane except possibly at $z = 0$ and $z = \infty$, which correspond to the ends of the period strip. But

$$|z q(z)| < 2\,|z| \cdot \left| \frac{\log z}{2\pi i} \right|,$$

which approaches 0 as $z \to 0$; hence $z q(z)$ is analytic and vanishes at $z = 0$. Likewise $q(z)/z$ is analytic and vanishes at $z = \infty$; $q(z)$ is therefore a single-valued analytic function which has no pole in the entire plane; accordingly it must be a constant,* namely 1, since $G(1) = \Gamma(1)$. Thus

* Cf. Osgood: *Lehrbuch der Funktionentheorie* I (2nd ed.), p. 310 (Riemann's Theorem) and p. 300 (Liouville's Theorem); or Burkhardt-Rasor: *Theory of Functions of a Complex Variable*, pp. 255, 230.

$$(69) \qquad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \qquad (u > 0).$$

To obtain an integral which shall be valid in the entire plane, let us take for the path of integration a loop circuit $L$ about $t = 0$, starting and ending at $t = \infty$ and enclosing the positive axis of reals;* this gives a solution of eq. (50). since $t^x v(t)$ or $t^x e^{-t}$ vanishes at both ends of the path. We will let $\arg t$ increase from $0$ to $2\pi$ on $L$. We may con-

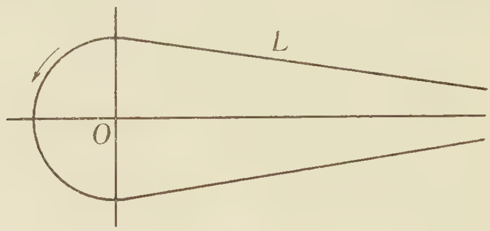


Fig. 2.

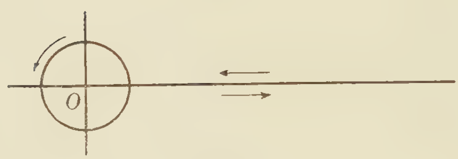


Fig. 3.

sider the path as consisting of the axis of reals from $\infty$ to $\varepsilon$, a small circle of radius $\varepsilon$ about $t = 0$, and the axis of reals from $\varepsilon$ back to $\infty$ (Fig. 3). If $u > 1$, the part of the integral contributed by the circle approaches $0$ as $\varepsilon \to 0$, and the first straight line integral approaches

$$\int_\infty^0 t^{x-1} e^{-t}\, dt = -\Gamma(x).$$

After a positive circuit of $t = 0$, the multiple-valued function $t^{x-1}$ is multiplied by $e^{2\pi i(x-1)} = e^{2\pi i x}$. Consequently we have in the limit for the return path

$$e^{2\pi i x} \int_0^\infty t^{x-1} e^{-t}\, dt = e^{2\pi i x}\, \Gamma(x).$$

* Any other ray in the right half plane could of course be used in place of the axis of reals.

Combining these results, we see that

$$(70)\qquad \int_L t^{x-1}\, e^{-t}\, dt = (e^{2\pi i x} - 1)\, \Gamma(x) = -\bar{\Gamma}(x).$$

This loop-integral represents a function analytic throughout the $x$-plane; $\bar{\Gamma}(x)$ is also analytic throughout the plane; therefore eq. (70), since it is true for the half plane $u > 1$, must be true for the entire $x$-plane. We obtain thus an integral formula for $\bar{\Gamma}(x)$, and also a second integral formula for $\Gamma(x)$, namely

$$(71)\qquad \Gamma(x) = \frac{1}{e^{2\pi i x} - 1} \int_L t^{x-1}\, e^{-t}\, dt,$$

which is valid throughout the plane except for positive integral values of $x$, where it is indeterminate.

If we set $t = 1/\tau$ in eq. (70), we get

$$(72)\qquad \bar{\Gamma}(x) = \int_K \tau^{-x-1}\, e^{-\frac{1}{\tau}}\, d\tau,$$

where $K$ is the contour into which $L$ is transformed by the substitution $t = 1/\tau$, namely a loop which starts from $\tau = 0$ in the right half plane, makes a positive circuit about $\tau = \infty$ (i. e., a negative circuit about $\tau = 0$), and returns to $\tau = 0$ (Fig. 4); on $K$ arg $\tau$ decreases from 0 to $-2\pi$. If we make this same substitution in eq. (69), we get the formula

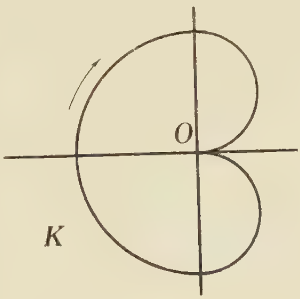

Fig. 4.

$$(73)\qquad \Gamma(x) = \int_0^\infty \tau^{-x-1}\, e^{-\frac{1}{\tau}}\, d\tau, \qquad (u > 0).$$

## § 3. Applications of the gamma function. Allied functions.

*A. Explicit solution of eq.* (38). We saw at the beginning of § 2 that the general solution of the equation

$$(38)\qquad y(x+1) - r(x)\, y(x) = 0$$

can be expressed in terms of any solution of eq. (50). Since the latter equation is satisfied by $\Gamma(x)$ and $\bar{\Gamma}(x)$, two particular solutions of eq. (38) are

$$(74)\quad y_1(x) = c_0^x \frac{\prod\limits_{k=1}^{m} \Gamma(x-\alpha_k)}{\prod\limits_{k=1}^{n} \Gamma(x-\beta_k)}, \quad y_2(x) = c_0^x \frac{\prod\limits_{k=1}^{m} \bar{\Gamma}(x-\alpha_k)}{\prod\limits_{k=1}^{n} \bar{\Gamma}(x-\beta_k)}.$$

Replacing $\Gamma$ and $\bar{\Gamma}$ by their asymptotic forms and simplifying [cf. (43)], we find that $y_1(x) \sim S(x)$ [eq. (44)] in the sector $-\pi < \arg x < \pi$ and $y_2(x) \sim S(x)$ in the sector $0 < \arg x < 2\pi$; $y_1(x)$ and $y_2(x)$ are therefore the first and second principal solutions of eq. (38).

These solutions are connected by the relation $y_2(x) = p(x)\, y_1(x)$, where $p(x)$ is a periodic function; its form can be determined directly by means of eq. (57), namely

$$(75)\qquad p(x) = \frac{y_2(x)}{y_1(x)} = \frac{\prod\limits_{k=1}^{m} \left(1 - e^{2\pi i(x-\alpha_k)}\right)}{\prod\limits_{k=1}^{n} \left(1 - e^{2\pi i(x-\beta_k)}\right)}.$$

*B. Summation of rational functions. The function* $\Psi(x)$. We considered in § 2, Chap. I, some examples of sums, and observed that the sum of a polynomial is another polynomial of the next higher degree; in fact, we have

$$(76)\quad \mathfrak{S}1 = x,\ \mathfrak{S}x = \binom{x}{2},\ \mathfrak{S}\binom{x}{2} = \binom{x}{3}, \cdots, \mathfrak{S}\binom{x}{n} = \binom{x}{n+1}.$$

Any polynomial $P(x)$ can be expressed linearly in terms of $1,\ x,\ \binom{x}{2}, \cdots$, since these are polynomials of degrees $0, 1, 2, \cdots$ respectively; hence $\mathfrak{S}\, P(x)$ is the same linear function of $x$, $\binom{x}{2}, \binom{x}{3}, \cdots$.

Consider now any rational function $r(x) = P(x) + Q(x)$, where $P(x)$ is a polynomial and $Q(x)$ a fraction whose numerator is of lower degree than its denominator. $Q(x)$ can be broken up into partial fractions of the form $c/(x-\alpha)^m$, so we will seek an analytic expression for $\mathop{\mathsf{S}} \dfrac{1}{(x-\alpha)^m}$.

From the equation $\Gamma(x+1) = x\Gamma(x)$ we obtain by logarithmic differentiation

$$\frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{1}{x} + \frac{\Gamma'(x)}{\Gamma(x)};$$

or, if we denote by $\Psi(x)$ the logarithmic derivative of $\Gamma(x)$,

$$\Psi(x+1) = \frac{1}{x} + \Psi(x). \tag{77}$$

Thus $\Psi(x)$ satisfies the difference equation

$$y(x+1) - y(x) = \frac{1}{x}, \tag{78}$$

which is of the form (6). Hence by the definition of summation we may write

$$\mathop{\mathsf{S}} \frac{1}{x} = \Psi(x).$$

Differentiating eq. (77), we have

$$\frac{d}{dx}\Psi(x+1) = -\frac{1}{x^2} + \frac{d}{dx}\Psi(x),$$

whence

$$\mathop{\mathsf{S}} \left(-\frac{1}{x^2}\right) = \frac{d}{dx}\Psi(x);$$

similarly, by repeated differentiation, we have

$$\mathop{\mathsf{S}} \frac{(-1)^m m!}{x^{m+1}} = \frac{d^m}{dx^m}\Psi(x),$$

from which we infer directly that

$$\mathop{\mathsf{S}} \frac{1}{(x-\alpha)^m} = \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dx^{m-1}} \Psi(x-\alpha). \tag{79}$$

If we denote by $\Psi(x)$ the function $\Gamma'(x)/\Gamma(x)$, which also satisfies eq. (78), we obtain in the same way another evaluation of this sum, namely

$$\mathop{\mathbf{S}} \frac{1}{(x-\alpha)^m} = \frac{(-1)^{m-1}}{(m-1)!} \cdot \frac{d^{m-1}}{dx^{m-1}} \Psi(x-\alpha). \tag{80}$$

We are accordingly able, by means of formulas (76) and (79) or (80), to express analytically the sum of any rational function. Furthermore, since $\log \Gamma(x)$ satisfies the difference equation

$$y(x+1) - y(x) = \log x,$$

we see that

$$\mathop{\mathbf{S}} \log x = \log \Gamma(x).$$

Since both $\Psi(x)$ and $\Psi(x)$ are solutions of eq. (78), their difference must be a periodic function, whose form can be obtained from eq. (57) by logarithmic differentiation, namely

$$\Psi(x) = \frac{2\pi i}{1 - e^{-2\pi i x}} + \Psi(x). \tag{81}$$

From Euler's relation (58) we obtain the corresponding formula for $\Psi(x)$:

$$\Psi(1-x) - \Psi(x) = \pi \cot \pi x.$$

C. *Asymptotic forms of $\Psi(x)$ and $\Phi(x, 1, k)$.* If in theorem V of § 6, Chap. I, we put

$$\varphi(x) = \log(1+x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \cdots,$$

$$f(x) = \frac{\Gamma(x)}{x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi}} - 1,$$

$$S(x) = \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51\,840x^3} - \cdots \quad \text{[see (64)]},$$

it follows that

$$\varphi(f(x)) = \log \frac{\Gamma(x)}{x^{x-\frac{1}{2}} e^{-x} \sqrt{2\pi}}$$
$$\sim S(x) - \frac{1}{2}[S(x)]^2 + \frac{1}{3}[S(x)]^3 - \cdots,$$

$$\log \Gamma(x) - \left(x - \frac{1}{2}\right)\log x + x - \frac{1}{2}\log 2\pi \tag{82}$$
$$\sim \frac{1}{12x} - \frac{1}{360x^3} + \cdots;$$

this holds in the sector $-\pi < \arg x < \pi$. By differentiation (cf. theorem IV, § 6, Chap. I),

$$\Psi(x) - \log x \sim -\frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \cdots; \tag{83}$$

$$\frac{d}{dx}\Psi(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \cdots \tag{84}$$

in the sector $-\pi < \arg(x-a) < \pi$, where $a$ is an arbitrary positive real number. These same formulas are true for $\Psi(x)$ in the sector $0 < \arg(x+a) < 2\pi$.

We are now in a position to determine the asymptotic form of the function

$$\Phi(x, r, k) = \frac{1}{x^k} + \frac{r}{(x+1)^k} + \frac{r^2}{(x+2)^k} + \cdots$$

for the case $r = 1$, $k \geqq 2$, which was not considered in our discussion in § 7, Chap. I. Take first the case $k = 2$. The equation (27) satisfied by $\Phi(x, 1, 2)$ is

$$y(x+1) - y(x) = -\frac{1}{x^2};$$

but this equation is also satisfied by $\frac{d}{dx}\Psi(x)$, so the difference $\frac{d}{dx}\Psi(x) - \Phi(x, 1, 2)$ must be a periodic function. Let $x \to \infty$ along a line parallel to the positive axis of reals; by (84) and by (33) both these functions and hence their

difference approach 0. The difference is therefore identically zero, so

$$(85)\quad \frac{d}{dx}\Psi(x) = \Phi(x,1,2) = \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \cdots;$$

the asymptotic form is given by (84). Likewise we find that

$$\frac{d^2}{dx^2}\Psi(x) = -2\,\Phi(x,1,3) = -\frac{2}{x^3} - \frac{2}{(x+1)^3} - \cdots.$$

etc. These results enable us to determine the asymptotic form of $\Phi(x, 1, k)$ for any $k \geqq 2$.

From eq. (85) we obtain by integration

$$\Psi(x) - \Psi(1) = -\left(\frac{1}{x} - 1\right) - \left(\frac{1}{x+1} - \frac{1}{2}\right) - \left(\frac{1}{x+2} - \frac{1}{3}\right) - \cdots.$$

But by eq. (55)

$$\log\Gamma(x) = -Cx - \log x + \sum_{n=1}^{\infty}\left(\frac{x}{n} + \log\frac{n}{x+n}\right),$$

where $C$ is Euler's constant; differentiating and setting $x = 1$, we have

$$\Psi(1) = -C - 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = -C.$$

$$(86)\quad \therefore \Psi(x) = -C - \sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right).$$

It is easily verified that this series converges uniformly in the neighborhood of every point except $x = 0, -1, -2, \cdots$, where $\Psi(x)$ has poles of the first order with residue $-1$.

*D. The beta function.* The function

$$(87)\qquad B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)},$$

which is known as the *beta function*, is important in the study of certain difference equations. Most of its fundamental

properties can be deduced at once from its definition; e. g., $B(y, x) = B(x, y)$,

$$(88)\qquad \begin{cases} B(x+1, y) = \dfrac{x}{x+y} B(x, y), \\ B(x, y+1) = \dfrac{y}{x+y} B(x, y). \end{cases}$$

The beta function can be expressed as a definite integral, namely

$$(89)\qquad B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt,$$

which is valid for $R(x) > 0$, $R(y) > 0$.* To prove this,† let $I(x, y)$ denote the integral. By integration by parts,

$$xI(x, y+1) = yI(x+1, y);$$

also we have

$$\begin{aligned} I(x, y+1) &= \int_0^1 t^{x-1}(1-t)^{y-1}(1-t)\, dt \\ &= I(x, y) - I(x+1, y). \end{aligned}$$

From these two equations we see that

$$I(x+1, y) = \frac{x}{x+y} I(x, y), \quad I(x, y+1) = \frac{y}{x+y} I(x, y).$$

Comparison with (88) shows that $I(x, y)$ and $B(x, y)$, regarded either as functions of $x$ or as functions of $y$, satisfy the same homogeneous difference equation. Their ratio $p(x, y) = I(x, y)/B(x, y)$ must accordingly be a function which is periodic in both $x$ and $y$.

Consider the behavior of $p(x, y)$ in the period strips $1 \leq R(x) < 2$, $1 \leq R(y) < 2$; it is analytic in both variables throughout the finite parts of the strips, since both $I(x, y)$ and $B(x, y)$ are analytic there and $B(x, y)$ does not vanish. Give $y$ a fixed value $y_0$ in its strip; then

* $R(x)$ and $R(y)$ denote the real parts of $x$ and $y$.

† Cf. Birkhoff: *Bull. Amer. Math. Soc.*, (2), 20 (1913), pp. 9–10.

$$|I(x, y_0)| \leqq \int_0^1 t^{R(x)-1}(1-t)^{R(y_0)-1}\,dt \leqq 1,$$

and for large values of $x$ in its strip, by (65),

$$|B(x, y_0)| = |\Gamma(y_0)| \cdot \left|\frac{\Gamma(x)}{\Gamma(x+y_0)}\right| > k\,|x^{-y_0}|,$$

where $k$ is a suitably chosen positive constant. Hence as $x \to \infty$ at either end of its strip

$$p(x, y_0) < \frac{1}{k}\, x^{y_0}.$$

If now we set $p(x, y_0) = q(z)$, where $z = e^{2\pi i x}$, we see as in the corresponding argument for $\Gamma(x)$ in § 2 that $q(z)$ is a single-valued analytic function which remains finite in the entire plane, i. e., it is a constant. By exactly the same reasoning, if we give $x$ a fixed value $x_0$ in its strip, the function $p(x_0, y)$ is a constant. Since $B(1,1) = I(1,1) = 1$, we must have $p(x, y) = 1$ identically; this proves eq. (89) for $R(x) > 0$, $R(y) > 0$.

The integral (89) is sometimes called the *first Eulerian integral*, while the integral for $\Gamma(x)$ in eq. (69) is called the *second Eulerian integral.*

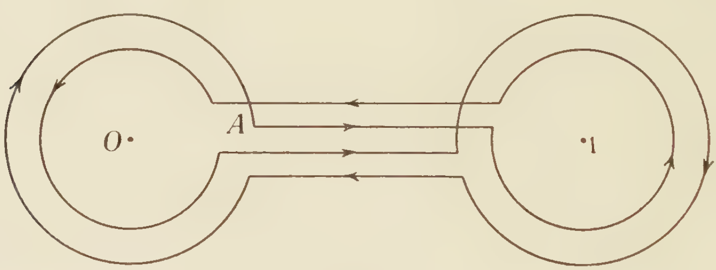


Fig. 5.

We can obtain an integral expression for $B(x, y)$ which is valid for all values of $x$ and $y$ by using as the path of integration instead of the straight line from 0 to 1 a double loop contour $I$ which starts at a point $A$, makes a positive

circuit about $t = 1$, then a positive circuit about $t = 0$, then a negative circuit about $t = 1$, and finally a negative circuit about $t = 0$, returning to its starting point. We may regard the contour as consisting of the axis of reals from $\varepsilon$ to $1-\varepsilon$, traversed twice in each direction, and small circles of radius $\varepsilon$ about 0 and 1, each traversed once in each direction. Let $A$ be the point $\varepsilon$, and along the first rectilinear portion of the path take $\arg t = \arg (1-t) = 0$; the integrand in (89) is multiplied in succession by $e^{2\pi i y}$, $e^{2\pi i x}$, $e^{-2\pi i y}$, and $e^{-2\pi i x}$ after traversing the circles in the order indicated, and thus returns to its initial value when the contour is completed. If the real parts of $x$ and $y$ are greater than 1, we can let $\varepsilon \to 0$; the part of the integral contributed by the circles then approaches 0, and we obtain in the limit

$$(1 - e^{2\pi i y} + e^{2\pi i (x+y)} - e^{2\pi i x}) \int_0^1 t^{x-1} (1-t)^{y-1} \, dt.$$

Hence for $R(x) > 1$, $R(y) > 1$ we have

$$\int_A t^{x-1} (1-t)^{y-1} \, dt = (1 - e^{2\pi i x})(1 - e^{2\pi i y}) \, B(x, y). \tag{90}$$

But both sides of this equation are analytic for all values of $x$ and $y$, and hence the equation is true in general.*

The functions $\Psi(x)$ and $B(x, y)$ treated in this section and $P_\varrho(x)$, $Q_\varrho(x)$ in the following section are examples of a numerous class of interesting functions which are closely related to the gamma function. Many treatments of the gamma function, indeed, are based on a preliminary study of $\Psi(x)$. For a more detailed discussion of these and other functions, most of which satisfy simple difference equations, the reader is referred to Nielsen's *Handbuch der Theorie der Gammafunktion*, and to the works cited in the "Literaturverzeichnis" of the latter.

## § 4. The general equation of the first order.

The general linear difference equation of the first order with rational coefficients

* Osgood: *Lehrbuch der Funktionentheorie* I (2nd ed.), p. 319.

$$(91) \qquad y(x+1) - r_1(x)y(x) = r_2(x)$$

has by eq. (20) the solution

$$(92) \qquad y(x) = y_1(x) \mathop{\mathrm{S}} \frac{r_2(x)}{y_1(x+1)},$$

where $y_1(x)$ is a solution of the associated homogeneous equation

$$(93) \qquad y(x+1) - r_1(x)y(x) = 0.$$

The most general solution of eq. (91) may by eq. (21) be written in the form

$$y(x) = \eta(x) + p(x)y_1(x),$$

where $\eta(x)$ is a particular solution and $p(x)$ is an arbitrary periodic function.

From eq. (92) we obtain the two symbolic solutions

$$y_r(x) = y_1(x)\left[-\frac{r_2(x)}{y_1(x+1)} - \frac{r_2(x+1)}{y_1(x+2)} - \cdots\right],$$

$$y_l(x) = y_1(x)\left[\frac{r_2(x-1)}{y_1(x)} + \frac{r_2(x-2)}{y_1(x-1)} + \cdots\right],$$

which satisfy eq. (91) formally. If we eliminate $y_1(x)$ with the aid of (93), these take the simpler form

$$(94) \quad \begin{cases} y_r(x) = -\dfrac{r_2(x)}{r_1(x)} - \dfrac{r_2(x+1)}{r_1(x)r_1(x+1)} \\ \qquad\qquad - \dfrac{r_2(x+2)}{r_1(x)r_1(x+1)r_1(x+2)} - \cdots, \\ y_l(x) = r_2(x-1) + r_1(x-1)r_2(x-2) \\ \qquad\qquad + r_1(x-1)r_1(x-2)r_2(x-3) + \cdots. \end{cases}$$

As in the case of the homogeneous equation (38), a power series in $1/x$ can be found which satisfies eq. (91) formally. If we write

$$r_1(x) = x^{\mu}\left(c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots\right), \qquad (c_0 \neq 0),$$

$$r_2(x) = x^{\nu}\left(d_0 + \frac{d_1}{x} + \frac{d_2}{x^2} + \cdots\right), \qquad (d_0 \neq 0),$$

this series has the form

$$x^{\nu-\mu}\left(s_0 + \frac{s_1}{x} + \frac{s_2}{x^2} + \cdots\right)$$

if $\mu > 0$, and

$$x^{\nu}\left(s_0 + \frac{s_1}{x} + \frac{s_2}{x^2} + \cdots\right)$$

if $\mu \leqq 0$.* The coefficients may be determined in any particular case by direct substitution.

With the aid of the symbolic solutions (94) and these series it is possible to prove the existence and study the properties of analytic solutions of eq. (91). The detailed discussion is rather long, however, and since it presents few new points of interest, we omit it.† Any solution of eq. (91), as indicated at the end of § 5, Chap. I, satisfies a homogeneous equation of the second order with rational coefficients, which can be treated by the methods of Chaps. III and IV.

Among the most interesting of the functions which satisfy non-homogeneous difference equations of the first order are *Prym's functions*

$$P(x) = \int_0^1 t^{x-1} e^{-t}\, dt, \qquad Q(x) = \int_1^\infty t^{x-1} e^{-t}\, dt;$$

these are special cases for $\varrho = 1$ of the more general functions

$$(95) \quad P_\varrho(x) = \int_0^\varrho t^{x-1} e^{-t}\, dt, \qquad Q_\varrho(x) = \int_\varrho^\infty t^{x-1} e^{-t}\, dt,$$

where $\varrho$ is any constant, real or complex. The first integral is valid for $x > 0$; the second is valid for all values of $x$,

* The case $\mu = 0$, $c_0 = 1$ is an exception; for this we have in general the factor $x^{\nu+1}$ instead of $x^{\nu}$.

† Cf. K. P. Williams: *Trans. Amer. Math. Soc.*, 14 (1913), pp. 209–240.

provided the path of integration goes to $\infty$ in the right half plane. Comparison of these definitions with (69) shows at once that

$$P_\varrho(x) + Q_\varrho(x) = \Gamma(x);$$

for this reason $P_\varrho(x)$ and $Q_\varrho(x)$ are sometimes called *incomplete gamma functions*. They satisfy the respective difference equations

$$P_\varrho(x+1) - xP_\varrho(x) = -e^{-\varrho}\varrho^x,$$
$$Q_\varrho(x+1) - xQ_\varrho(x) = e^{-\varrho}\varrho^x,$$

as may be verified by substitution and integration by parts; both (as well as $\Gamma(x)$) are solutions of the homogeneous equation

(96) $\quad y(x+2) - (x+\varrho+1)\,y(x+1) + \varrho x\,y(x) = 0$

(cf. end of § 5, Chap. I and § 2, Chap. IV).

Integrating $P_\varrho(x)$ by parts $n+1$ times, we find that

$$P_\varrho(x) = \varrho^x e^{-\varrho}\left[\frac{1}{x} + \frac{\varrho}{x(x+1)} + \cdots + \frac{\varrho^n}{x(x+1)\cdots(x+n)}\right]$$
$$+ \frac{1}{x(x+1)\cdots(x+n)}\int_0^{\varrho} t^{x+n}e^{-t}\,dt.$$

If we set $t = \varrho\tau$ in the last term, the integral becomes

$$\varrho^{x+n}\int_0^1 \tau^{x+n}e^{-\varrho\tau}\varrho\,d\tau,$$

which is less in absolute value than

$$\left|\varrho^{x+n}\int_0^1 e^{-\varrho\tau}\varrho\,d\tau\right| = \left|\varrho^{x+n}(1-e^{-\varrho})\right|.$$

Hence the remainder term approaches 0 uniformly as $n \to \infty$ except near $x = 0, -1, -2, \cdots$, and we have in the limit

(97) $\quad P_\varrho(x) = \varrho^x e^{-\varrho}\sum_{n=0}^{\infty}\frac{\varrho^n}{x(x+1)\cdots(x+n)};$

This series converges uniformly in any closed region which does not contain any of the points $x = 0, -1, -2, \cdots$, and so represents a function analytic throughout the plane except for poles of the first order at these points. Eq. (97) may be taken as the definition of $P_\varrho(x)$ in the left half plane, where the integral definition is not valid.

This expression for $P_\varrho(x)$ gives us an example of what are known as *factorial series*, i. e., series of the type

$$
(98) \qquad \begin{aligned} &\Gamma(x) \sum_{n=0}^{\infty} \frac{n!\, a_n}{\Gamma(x+n)} \\ &= a_0 + \frac{a_1}{x} + \frac{2!\, a_2}{x(x+1)} + \frac{3!\, a_3}{x(x+1)(x+2)} + \cdots, \end{aligned}
$$

which are great importance in the theory of difference equations, and have important applications elsewhere.*

A partial fraction series for $P_\varrho(x)$ is obtained if we expand $e^{-t}$ in (95) and integrate term by term, namely

$$P_\varrho(x) = \varrho^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\varrho^n}{x+n}.$$

This shows at once that the residue of $P_\varrho(x)$ at the pole $x = -k$ is $(-1)^k/k!$, which is the same as that of $\Gamma(x)$; hence $Q_\varrho(x)$, which is equal to $\Gamma(x) - P_\varrho(x)$, is analytic for all finite values of $x$, and so is an entire function.

It was proved by Hölder† that the gamma function cannot satisfy an algebraic differential equation, and this result was extended by Barnes‡ to the solutions of eq. (91). The transcendental functions defined by difference equations are evidently of an essentially different type from those defined by differential equations.

* For an introductory treatment of factorial series see Landau: *Sitzungsberichte Akad. München (math.-phys.)*, 36 (1906), pp. 151–218.

† *Math. Annalen*, 28 (1886), pp. 1–13. Cf. simplified proofs by F. Hausdorff and A. Ostrowski in *Math. Annalen*, 94 (1925), pp. 244–251.

‡ *Proc. London Math. Soc.*, (2), 2 (1904), pp. 280–292.

## Exercises.

1. Locate the poles and zeros of the principal solutions of the equations

$$y(x+1) = \left(1 \pm \frac{1}{x^2}\right) y(x).$$

2. Obtain the explicit form of the periodic function $p(x) = h(x)/g(x)$ for the equations of ex. 1.

3. Locate the poles and zeros of the principal solutions of the equation

$$y(x+1) = \frac{x+1}{x-2}\, y(x),$$

and obtain directly the relation between the solutions.

4. Verify the results obtained in ex. 3 by solving the equation in terms of the gamma function.

5. Prove that a necessary and sufficient condition for the convergence of the formal power series solution of the equation

$$y(x+1) = \left[1 + \frac{\theta(x)}{x^2}\right] y(x),$$

where $|\theta(x)| < M$ for $x > R$, is that $y_l(x) = y_r(x)$.

6. Evaluate the sums

$$\mathop{\mathsf{S}} \frac{x^3}{x^2-1}; \qquad \mathop{\mathsf{S}} \frac{(x-1)^2}{x^3}; \qquad \mathop{\mathsf{S}} \log\left(1+\frac{1}{x}\right)^x.$$

7. Obtain the approximate value to four decimal places of $\Phi(x, 1, 2)$ for $x = 100$.

8. Derive a series for $\overline{\Psi}(x)$ similar to (86).

9. Prove that

$$\Gamma(x) = \sqrt{B\left(x, \frac{1}{2}\right)} \sqrt{\frac{\Gamma(2x)}{2^{2x-1}}}.$$

10. If $R(x) > 0$, $R(y) > 0$, prove that

$$\text{(a)} \quad \int_0^\infty t^{x-1} e^{-t^2} dt \qquad = \frac{1}{2}\, \Gamma\left(\frac{x}{2}\right);$$

$$\text{(b)} \quad \int_0^a t^{x-1} (a-t)^{y-1} dt = a^{x+y-1} B(x, y).$$

11. Prove that if $R(x) > 0$,

$$\frac{1}{2\pi i}\int_l t^{x-1}(t-1)^{y-1}\,dt = \frac{\Gamma(x)}{\Gamma(1-y)\,\Gamma(x+y)},$$

where $l$ is a loop circuit starting and ending at $t=0$ and passing around $t=1$ in the positive direction.

12. Prove that if $R(x+y) < 1$,

$$\frac{1}{2\pi i}\int_L t^{x-1}(t-1)^{y-1}\,dt = \frac{e^{\pi i y}\,\Gamma(1-x-y)}{\Gamma(1-x)\,\Gamma(1-y)},$$

where $L$ is a loop starting and ending at $t=\infty$ on the positive axis of reals and passing around $t=1$ in the positive direction.

13. Find the asymptotic form of the factorial series

$$\frac{1}{x}+\frac{1}{x(x+1)}+\frac{1}{x(x+1)(x+2)}+\cdots$$

and hence of Prym's function $P(x)$.

14. Obtain the power series which satisfy formally the equations

(a) $y(x+1) = \dfrac{x+1}{x^2}\,y(x)$;

(b) $y(x+1)-y(x) = \dfrac{1}{x^2}$;

(c) $y(x+1)-x\,y(x) = -\dfrac{1}{e}$.

15. Investigate the convergence of the symbolic solutions (94).

16. Solve the equations

(a) $y(x+1)-\left(1-\dfrac{1}{x^2}\right)y(x) = x+1$;

(b) $x\,y(x+1)-(x+1)\,y(x) = 1$.

17. Find a definite integral solution of

$$y(x+1)-2y(x) = \frac{1}{x}.$$

$\left(\text{Hint: } \dfrac{1}{x} = \displaystyle\int_0^1 t^{x-1}\,dt\right)$. Express this solution as a factorial series and as a partial fraction series, and discuss their convergence.

# Chapter III.

## The hypergeometric equation: general case.

### § 1. The hypergeometric difference equation.

Next to equations with constant coefficients, which we studied in § 4, Chap. I, the simplest linear homogeneous difference equations of higher than the first order are those with linear coefficients. The second order equation of this type, namely

$$(99)\qquad (a_2 x + b_2)\, y(x+2) + (a_1 x + b_1)\, y(x+1) + (a_0 x + b_0)\, y(x) = 0,$$

is called the *hypergeometric difference equation*, because, as we shall see in § 7, its solutions can be expressed in terms of the hypergeometric series

$$(100)\ F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{1\cdot\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\,\gamma(\gamma+1)} z^2 + \cdots.$$

The theory of this equation presents most of the interesting features of that of the general homogeneous linear difference equation of the $n$th order with rational coefficients, but in a less abstract form, since explicit formulas can be obtained throughout. Accordingly the remainder of this book will be devoted to a detailed study of the hypergeometric equation.

An important rôle in the theory is played by the roots $\varrho_1$ and $\varrho_2$ of the *characteristic equation*

$$(101)\qquad a_2 \varrho^2 + a_1 \varrho + a_0 = 0.$$

We shall consider first the "general case", in which $\varrho_1$ and $\varrho_2$ are finite, distinct from each other, and different from zero; in terms of the coefficients, this means that none of the three

numbers $a_2$, $a_0$, $a_1^2 - 4a_0 a_2$ is zero. We will choose the subscripts so that $|\varrho_1| > |\varrho_2|$. The "irregular cases", in which one or more of these conditions is not satisfied, present certain additional difficulties; these will be treated in Chap. IV.

In order to get eq. (99) into a more convenient form, we will define three new constants $\beta_1$, $\beta_2$, $\beta_3$ by the equations

$$\begin{aligned} \beta_1 + \beta_2 + \beta_3 + 2 &= \frac{b_2}{a_2}, \\ \beta_1 \varrho_2 + \beta_2 \varrho_1 + (\beta_3 + 1)(\varrho_1 + \varrho_2) &= -\frac{b_1}{a_2}, \\ \varrho_1 \varrho_2 \beta_3 &= \frac{b_0}{a_2}; \end{aligned}$$

these have a unique solution under our hypotheses, since the determinant of the coefficients is $\varrho_1 \varrho_2 (\varrho_1 - \varrho_2) \neq 0$. From eq. (101) we have the relations

$$-(\varrho_1 + \varrho_2) = \frac{a_1}{a_2}, \qquad \varrho_1 \varrho_2 = \frac{a_0}{a_2};$$

eq. (99) can accordingly be written in the form

$$(102) \begin{cases} (x + \beta_1 + \beta_2 + \beta_3 + 2)\, y(x+2) \\ \qquad - [(\varrho_1 + \varrho_2)(x + \beta_3 + 1) + \beta_1 \varrho_2 + \beta_2 \varrho_1]\, y(x+1) \\ \qquad + \varrho_1 \varrho_2 (x + \beta_3)\, y(x) = 0. \end{cases}$$

Set $x + \beta_3 = x'$ and $y(x' - \beta_3) = f(x')$; then, dropping the primes, we see that $f(x)$ satisfies the equation

$$(103) \begin{cases} (x + \beta_1 + \beta_2 + 2)\, y(x+2) \\ \qquad - [(\varrho_1 + \varrho_2)(x+1) + \beta_1 \varrho_2 + \beta_2 \varrho_1]\, y(x+1) \\ \qquad + \varrho_1 \varrho_2\, x y(x) = 0, \end{cases}$$

which may be regarded as the normal form of eq. (99) for the general case.

## § 2. Formal power series solutions.

Eq. (103) is satisfied formally by two power series in $1/x$, multiplied by certain exponential factors; we cannot have a factor of the form $x^{ax}$, since the coefficients are all of the same degree; so we will set

$$(104)\qquad y(x) = c^x x^k \left(s + \frac{s'}{x} + \frac{s''}{x^2} + \dots\right)$$

in eq. (103). Dividing by $c^x x^k$, we have

$$c^2(x + \mathfrak{b}_1 + \mathfrak{b}_2 + 2)\left(1 + \frac{2}{x}\right)^k \left[s + \frac{s'}{x}\left(1 + \frac{2}{x}\right)^{-1} + \dots\right]$$

$$- c[(\alpha_1 + \alpha_2)(x + 1) + \mathfrak{b}_1\alpha_2 + \mathfrak{b}_2\alpha_1]\left(1 + \frac{1}{x}\right)^k$$

$$[\text{illegible}]$$

$$+ \alpha_1\alpha_2 x\left(s + \frac{s'}{x} + \frac{s''}{x^2} + \dots\right) = 0.$$

Expanding in powers of $1/x$, and equating the coefficients of successive powers to 0, we find that

$$(105)\qquad \rho^2 - c(\alpha_1 + \alpha_2) + \alpha_1\alpha_2 = 0,$$

$$[\text{illegible}]$$

$$- c k(\alpha_1 + \alpha_2)s - c(\alpha_1 + \alpha_2)s + \alpha_1\alpha_2 s' = 0,$$

etc., whence $c = \alpha_1$ or $\alpha_2$, $k = -\mathfrak{b}_1 - 1$ or $-\mathfrak{b}_2 - 1$,

$$\frac{s'}{s} = (\mathfrak{b}_1 + 1)\left(\frac{\mathfrak{b}_2\alpha_1}{\alpha_2 - \alpha_1} - \frac{\mathfrak{b}_1}{2}\right), \qquad \frac{s'}{s} = (\mathfrak{b}_2 + 1)\left(\frac{\mathfrak{b}_1\alpha_2}{\alpha_1 - \alpha_2} - \frac{\mathfrak{b}_2}{2}\right),$$

etc., where $s'/s$ and $s'/s$ are the two values of $s'/s$; $s_1$ and $s_2$ are of course [illegible]. To prove that all the coefficients can be obtained in this way, assume that $s'/s$, $s''/s$, ..., $s^{(k-1)}/s$ have been determined for both values. Equating to 0 the coefficient of $1/x^k$, we have

$$c^2(s^{(k+1)} - 2ks^{(k)}) + c^2(\mathfrak{b}_1 + \mathfrak{b}_2 + 2 + 2k)s^{(k)}$$

$$- c(\alpha_1 + \alpha_2)(s^{(k+1)} - ks^{(k)}) - c(\alpha_1 + \alpha_2 + \mathfrak{b}_1\alpha_2 + \mathfrak{b}_2\alpha_1)s^{(k)}$$

$$- ck(\alpha_1 + \alpha_2)s^{(k)} + \alpha_1\alpha_2 s^{(k+1)} + As = 0,$$

where $A$ involves only quantities which have been already determined. The terms in $s^{(k+1)}$ vanish, by eq. (105). If we

we first set $\rho = \rho_1$, $\sigma = \beta_1 - 1$, $s = s_1$ and then $\rho = \rho_2$, $\sigma = \beta_2 - 1$, $s = s_2$, we obtain the two equations

$$k\rho_1(\rho_2 - \rho_1)s_1^{(k)} + A_1 s_1 \cdots = 0,$$
$$k\rho_2(\rho_1 - \rho_2)s_2^{(k)} + A_2 s_2 \cdots = 0,$$

which serve to determine $s_1^{(k)}/s_1$ and $s_2^{(k)}/s_2$ uniquely. Hence every coefficient can be determined in terms of previously found quantities, and we thus obtain the two series

$$(104)\quad \begin{cases} s_1(x) = \rho_1^x x^{\beta_1 - 1}\left(s_1 + \dfrac{s_1'}{x} + \dfrac{s_1''}{x^2} + \cdots\right), \\[2ex] s_2(x) = \rho_2^x x^{\beta_2 - 1}\left(s_2 + \dfrac{s_2'}{x} + \dfrac{s_2''}{x^2} + \cdots\right), \end{cases}$$

which, from the way they were derived, must satisfy formally eq. (103). In general they are divergent, and so do not give us analytic solutions. They play a very important part in the theory, however, and, as we shall see, represent the principal solutions asymptotically for large values of $x$ in certain sectors of the plane.

## § 3. The use of matrices.*

For the purpose of deriving existence theorems for the analytic solutions of a difference equation, it is in many respects preferable to deal with a system of two equations of the first order rather than a single equation of the second order. This permits the use of matrices, which serve to bring out the analogies which exist between an equation of the first order and one of higher order.

A system equivalent to eq. (103) may be obtained in various ways; the most direct is to set $y(x) = y_1(x)$, $y(x+1) = y_2(x)$; then $y_1(x)$ and $y_2(x)$ satisfy the system

* An elementary knowledge of matrices, such as is given by Bôcher: *Introduction to Higher Algebra*, §§ 7, 21, 22, 25, is presupposed in what follows.

$$
(107)\begin{cases} y_1(x+1) = y_2(x), \\ y_2(x+1) = -\dfrac{\varrho_1\,\varrho_2\,x}{x+\beta_1+\beta_2+2}\,y_1(x) \\ \qquad\qquad + \dfrac{(\varrho_1+\varrho_2)(x+1)+\beta_1\,\varrho_2+\beta_2\,\varrho_1}{x+\beta_1+\beta_2+2}\,y_2(x). \end{cases}
$$

A more symmetrical system could be obtained, but this has the advantage of being closely connected with eq. (103): if we have any solution of (107), the first element $y_1(x) = y(x)$ gives us directly a solution of (103).

If $y_{11}(x)$, $y_{21}(x)$ and $y_{12}(x)$, $y_{22}(x)$ are any two linearly independent solutions of (107), the matrix

$$
Y(x) = \| y_{ij}(x) \| = \begin{Vmatrix} y_{11}(x) & y_{12}(x) \\ y_{21}(x) & y_{22}(x) \end{Vmatrix}
$$

is called a *matrix solution* of the system. The coefficients of the $y$'s in (107) form another matrix:

$$
R(x) = \begin{Vmatrix} 0 & 1 \\ -\dfrac{\varrho_1\,\varrho_2\,x}{x+\beta_1+\beta_2+2} & \dfrac{(\varrho_1+\varrho_2)(x+1)+\beta_1\,\varrho_2+\beta_2\,\varrho_1}{x+\beta_1+\beta_2+2} \end{Vmatrix}.
$$

The four equations satisfied by the elements of $Y(x)$ can be combined into a single matrix equation:

$$
(108) \qquad Y(x+1) = R(x)\,Y(x);
$$

this is equivalent to the system (107), as may be verified by expanding and equating corresponding elements on the two sides.

Let $P(x)$ be a matrix whose elements are periodic functions of period 1, and whose determinant is not identically 0; then $Y(x)\,P(x)$ is also a solution of (108); for by substitution we have

$$
Y(x+1)\,P(x+1) = R(x)\,Y(x)\,P(x);
$$

but $P(x+1) = P(x)$; hence, multiplying both sides on the right by the inverse matrix $P^{-1}(x)$, we get eq. (108) again. If the elements of $P(x)$ are *arbitrary* periodic functions (subject only to the condition that their determinant does not vanish identically), $Y(x)\,P(x)$ is called the *general matrix solution* of (108), since by choosing the periodic functions properly we can put any matrix solution in this form, as is evident if we expand $Y(x)\,P(x)$ into its elements (cf. § 3, Chap. I).

The matrix equation (108) is satisfied formally by a matrix of power series whose elements are readily obtained from (106), namely

$$S(x) = \begin{Vmatrix} s_{11}(x) & s_{12}(x) \\ s_{21}(x) & s_{22}(x) \end{Vmatrix} = \begin{Vmatrix} S_1(x) & S_2(x) \\ S_1(x+1) & S_2(x+1) \end{Vmatrix}$$

$$(109) \qquad = \begin{Vmatrix} \varrho_1^x x^{-\beta_1-1}\left(s_{11}+\dfrac{s'_{11}}{x}+\cdots\right) & \varrho_2^x x^{-\beta_2-1}\left(s_{12}+\dfrac{s'_{12}}{x}+\cdots\right) \\ \varrho_1^x x^{-\beta_1-1}\left(s_{21}+\dfrac{s'_{21}}{x}+\cdots\right) & \varrho_2^x x^{-\beta_2-1}\left(s_{22}+\dfrac{s'_{22}}{x}+\cdots\right) \end{Vmatrix},$$

where $s_{11} = s_1$, $s'_{11} = s'_1, \cdots$; $s_{21} = \varrho_1 s_1$, $s'_{21} = \varrho_1(s'_1 - \beta_1 s_1 - s_1)$, etc. The determinant of this matrix has the form

$$(110) \quad |S(x)| = (\varrho_1\varrho_2)^x\, x^{-\beta_1-\beta_2-2}\left(d + \frac{d'}{x} + \frac{d''}{x^2} + \cdots\right),$$

where $d = (\varrho_2 - \varrho_1)\, s_1 s_2$, etc. The inverse of $S(x)$ is

$$(111) \; S^{-1}(x) = \begin{Vmatrix} \varrho_1^{-x} x^{\beta_1+1}\left(\sigma_{11}+\dfrac{\sigma'_{11}}{x}+\cdots\right) & \varrho_1^{-x} x^{\beta_1+1}\left(\sigma_{12}+\dfrac{\sigma'_{12}}{x}+\cdots\right) \\ \varrho_2^{-x} x^{\beta_2+1}\left(\sigma_{21}+\dfrac{\sigma'_{21}}{x}+\cdots\right) & \varrho_2^{-x} x^{\beta_2+1}\left(\sigma_{22}+\dfrac{\sigma'_{22}}{x}+\cdots\right) \end{Vmatrix},$$

where

$$\sigma_{11} = \frac{\varrho_2}{(\varrho_2-\varrho_1)\,s_1}, \qquad \sigma_{12} = \frac{1}{(\varrho_1-\varrho_2)\,s_1},$$

$$\sigma_{21} = \frac{\varrho_1}{(\varrho_1-\varrho_2)\,s_2}, \qquad \sigma_{22} = \frac{1}{(\varrho_2-\varrho_1)\,s_2},$$

etc.

Let $T(x)$ denote the matrix obtained from $S(x)$ by using only the first $k$ terms of each series, and write

$$(112) \qquad T(x)\,T^{-1}(x+1) = B(x).$$

Since $S(x)$ satisfies eq. (108), we have $S(x)S^{-1}(x+1) = R^{-1}(x)$; by the definition of $T(x)$, $T^{-1}(x+1)$ has the same first $k$ terms as $S^{-1}(x+1)$; hence the elements of $B(x)$ differ from those of $R^{-1}(x)$ only in the terms after the $k$th, so we may write

$$R^{-1}(x) = B(x)\left[I + \frac{1}{x^k}\,C(x)\right],$$

where $I$ denotes the unit matrix and $C(x)$ is a matrix whose elements are of the form

$$c_{ij} + \frac{c'_{ij}}{x} + \frac{c''_{ij}}{x^2} + \cdots.$$

## § 4. First existence theorem.

The matrix equation (108) is satisfied formally by the two symbolic solutions

$$(113) \qquad \begin{cases} R^{-1}(x)\,R^{-1}(x+1)\,R^{-1}(x+2)\cdots, \\ R(x-1)\,R(x-2)\,R(x-3)\cdots. \end{cases}$$

as is seen by direct substitution. These infinite products are in general divergent, but by a suitable modification, as in § 1, Chap. II, convergent products can be obtained which yield analytic solutions. For this purpose we will form the products

$$(114) \quad \begin{cases} H_n(x) = R^{-1}(x)\,R^{-1}(x+1)\cdots R^{-1}(x+n)\,T(x+n+1), \\ G_n(x) = R(x-1)\,R(x-2)\cdots R(x-n)\,T(x-n). \end{cases}$$

THEOREM. *Both elements of the second column of $H_n(x)$ converge uniformly as $n \to \infty$ to limit functions $h_{12}(x)$, $h_{22}(x)$ which are analytic in the entire finite part of the plane except for poles, and form a solution of the system* (107). *The determinant of $H_n(x)$ also converges to a limit function $D(x)$*

*analytic except for poles. For large values of $x$ these limit functions are represented asymptotically by $s_{12}(x)$, $s_{22}(x)$, and $|S(x)|$ respectively in the sector $-\pi < \arg x < \pi$.*

*Similarly, both elements of the first column of $G_n(x)$ converge uniformly to limit functions $g_{11}(x)$, $g_{21}(x)$ which are analytic except for poles and form a second solution of* (107). *The determinant of $G_n(x)$ also converges uniformly to a limit function $\bar{D}(x)$ analytic except for poles. For large values of $x$ these limit functions are represented asymptotically by $s_{11}(x)$, $s_{21}(x)$, and $|S(x)|$ respectively in the sector $0 < \arg x < 2\pi$.*

We will write $H_n(x) = T(x)\, H_n'(x)$, where

$$\begin{aligned} H_n'(x) = {} & T^{-1}(x)\, R^{-1}(x)\, T(x+1) \\ & \times T^{-1}(x+1)\, R^{-1}(x+1)\, T(x+2) \cdots \\ & \times T^{-1}(x+n)\, R^{-1}(x+n)\, T(x+n+1). \end{aligned}$$

Thus $H_n'(x)$ is a product of matrices of the form

$$\begin{aligned} T^{-1}(x)\, R^{-1}(x)\, T(x+1) &= T^{-1}(x)\, B(x) \left[ I + \frac{1}{x^k}\, C(x) \right] T(x+1) \\ &= I + \frac{1}{x^k}\, T^{-1}(x+1)\, C(x)\, T(x+1), \end{aligned}$$

since by (112) $T^{-1}(x)\, B(x) = T^{-1}(x+1)$. Since the first $k$ terms of $T^{-1}(x+1)$ and $T(x+1)$ are the same as those of $S^{-1}(x+1)$ and $S(x+1)$, we see from (109) and (111) that

$$T^{-1}(x)\, R^{-1}(x)\, T(x+1)$$

$$(115) \qquad = I + \frac{1}{x^k} \left\| \begin{matrix} \psi_{11}(x) & \left(\frac{\varrho_2}{\varrho_1}\right)^x x^{\beta_1-\beta_2}\, \psi_{12}(x) \\ \left(\frac{\varrho_1}{\varrho_2}\right)^x x^{\beta_2-\beta_1}\, \psi_{21}(x) & \psi_{22}(x) \end{matrix} \right\|,$$

where the functions $\psi_{ij}(x)$ have the form

$$\psi_{ij} + \frac{\psi_{ij}'}{x} + \frac{\psi_{ij}''}{x^2} + \cdots.$$

Let us choose $k$ large enough so that the real parts of $\beta_1 - \beta_2 - k$ and $\beta_2 - \beta_1 - k$ are both $\leqq -2$; then the right hand side of (115) may be written

$$I+\frac{1}{x^\lambda}\left\|\begin{array}{cc} \varphi_{11}(x) & \left(\frac{\varrho_2}{\varrho_1}\right)^x \varphi_{12}(x) \\ \left(\frac{\varrho_1}{\varrho_2}\right)^x \varphi_{21}(x) & \varphi_{22}(x) \end{array}\right\| = I+\frac{1}{x^\lambda}\,\Theta(x),$$

where $\lambda$ is an integer $\geqq 2$ and the functions $\varphi_{ij}(x)$ remain finite as $x \to \infty$.

The matrix $H_n'(x)$ may now be written

$$\left[I+\frac{1}{x^\lambda}\,\Theta(x)\right]\left[I+\frac{1}{(x+1)^\lambda}\,\Theta(x+1)\right]\cdots$$
$$\times\left[I+\frac{1}{(x+n)^\lambda}\,\Theta(x+n)\right]$$

$$= I+\sum_{r=0}^{n}\frac{1}{(x+r)^\lambda}\,\Theta(x+r)$$
$$+\sum_{r=0}^{n-1}\sum_{s=r+1}^{n}\frac{1}{(x+r)^\lambda (x+s)^\lambda}\,\Theta(x+r)\,\Theta(x+s)+\cdots.$$

The $i$th element $(i=1, 2)$ of the second column is

$$\delta_{i2}+\sum_{r=0}^{n}\frac{1}{(x+r)^\lambda}\left(\frac{\varrho_2}{\varrho_i}\right)^{x+r}\varphi_{i2}(x+r)$$
$$(116)\qquad +\sum_{r=0}^{n-1}\sum_{s=r+1}^{n}\frac{1}{(x+r)^\lambda (x+s)^\lambda}\sum_{\gamma=1}^{2}\left(\frac{\varrho_\gamma}{\varrho_i}\right)^{x+r}\left(\frac{\varrho_2}{\varrho_\gamma}\right)^{x+s}$$
$$\times\,\varphi_{i\gamma}(x+r)\,\varphi_{\gamma 2}(x+s)+\cdots.$$

where $\delta_{i2}=0$ or $1$ according as $i=1$ or $2$. We need to prove that this converges uniformly as $n\to\infty$. Since the functions $\varphi_{ij}(x)$ remain finite, we can find constants $M$ and $R$ such that $|\varphi_{ij}(x)|<M$ for $|x|>R$. Hence if all the points $x, x+1, x+2, \cdots$ lie outside the circle $|x|=R$, i. e., if $x$ lies in the region $D$ of Fig. 1, consisting of the entire plane except the part within a distance $R$ of the negative axis of reals, the typical term in one of the sums in (116), namely

$$\frac{1}{(x+r)^\lambda (x+s)^\lambda\cdots(x+w)^\lambda}\left(\frac{\varrho_\gamma}{\varrho_i}\right)^{x+r}\left(\frac{\varrho_\delta}{\varrho_\gamma}\right)^{x+s}\cdots\left(\frac{\varrho_2}{\varrho_\eta}\right)^{x+w}$$
$$\times\,\varphi_{i\gamma}(x+r)\cdots\varphi_{\eta 2}(x+w)$$

$(\gamma, \delta, \cdots, \eta = 1 \text{ or } 2)$, is less in absolute value than

$$\left(\frac{\varrho_2}{\varrho_i}\right)^x \frac{M^l}{(x+r)^\lambda (x+s)^\lambda \cdots (x+w)^\lambda}$$

where $l$ is the number of integers in the sequence $r, s, \cdots, w$; this follows from the fact that

$$(117)\quad \left(\frac{\varrho_\gamma}{\varrho_i}\right)^{x+r}\left(\frac{\varrho_\delta}{\varrho_\gamma}\right)^{x+s}\cdots\left(\frac{\varrho_2}{\varrho_\eta}\right)^{x+w} = \left(\frac{\varrho_2}{\varrho_i}\right)^{x+r}\left(\frac{\varrho_2}{\varrho_\gamma}\right)^{s-r}\cdots\left(\frac{\varrho_2}{\varrho_\eta}\right)^{w-v} \leqq \left|\left(\frac{\varrho_2}{\varrho_i}\right)^x\right|,$$

since by hypothesis $|\varrho_2| < |\varrho_1|$, and $r, s, \cdots, w$ are an increasing set of positive integers. The series (116) is therefore less term by term in absolute value than

$$\left|\left(\frac{\varrho_2}{\varrho_i}\right)^x\right| \sum_{r=0}^{n} \frac{M}{|x+r|^\lambda} + \left|\left(\frac{\varrho_2}{\varrho_i}\right)^x\right| \sum_{r=0}^{n-1}\sum_{s=r+1}^{n} \frac{2M^2}{|x+r|^\lambda |x+s|^\lambda} + \cdots.$$

As $n \to \infty$, this approaches the limit

$$(118)\quad \frac{1}{2}\left|\left(\frac{\varrho_2}{\varrho_i}\right)^x\right| \left\{\left|\left(1+\frac{2M}{|x|^\lambda}\right)\left(1+\frac{2M}{|x+1|^\lambda}\right)\cdots\right| - 1\right\};$$

since $\lambda \geqq 2$, the infinite product here converges, so the elements of the second column of $H_n'(x)$ and hence of $H_n(x)$ converge uniformly to analytic functions in the neighborhood of any point of the region $D$. Moreover, since

$$H_n(x) = R^{-1}(x)\, R^{-1}(x+1) \cdots R^{-1}(x+m-1)\, H_{n-m}(x+m),$$

where $m$ may be taken so large that the point $x+m$ lies in $D$, we see that they converge also in the region excluded above; the only exceptions are the points $0, -1, -2, \cdots$, which are the poles of elements of $R^{-1}(x)$, $R^{-1}(x+1)$, $\cdots$.

The limit functions $h_{12}(x)$, $h_{22}(x)$ of the elements of the second column of $H_n(x)$ are therefore analytic at every finite point of the plane except 0, $-1$, $-2$, $\cdots$, where in general they have poles.

The functions $h_{12}(x)$, $h_{22}(x)$ thus obtained are independent of the value of $k$ chosen; for if $T'(x)$ is the matrix obtained from $S(x)$ by using a different value $k'$, we have for our product

$$\begin{aligned}\bar{H}_n(x) &= R^{-1}(x) R^{-1}(x+1) \cdots R^{-1}(x+n)\, T'(x+n+1) \\ &= R^{-1}(x) \cdots R^{-1}(x+n)\, T(x+n+1) \\ &\qquad\qquad \times T^{-1}(x+n+1)\, T'(x+n+1),\end{aligned}$$

whose second column converges to the same limit as that of $H_n(x)$, since

$$\lim_{n\to\infty} T^{-1}(x+n+1)\, T'(x+n+1) = I.$$

regardless of the values of $k$ and $k'$. They form a solution of the system (107), since by the definition of $H_n(x)$

$$H_n(x+1) = R(x)\, H_{n+1}(x),$$

a relation which remains true for the elements of the second column as we pass to the limit.

Consider now the behavior of $h_{12}(x)$ and $h_{22}(x)$ for large values of $x$. As we have seen, each element of the second column of $H_n'(x)$ remains less in absolute value than (118), which is itself less than

$$\delta_{i2} + \frac{1}{2} \left| \left(\frac{\varrho_2}{\varrho_i}\right)^x \right| \left( e^{\sum\limits_{r=0}^{\infty} \frac{2M}{|x+r|^\lambda}} - 1 \right).$$

By the inequalities (28) and (29), these elements can be expressed in the form

$$h_{i2}'(x) = \delta_{i2} + \left(\frac{\varrho_2}{\varrho_i}\right)^x \frac{M_i(x)}{x^{\lambda-1}} \quad \text{or} \quad \delta_{i2} + \left(\frac{\varrho_2}{\varrho_i}\right)^x \frac{M_i(x)}{x^{\lambda-1}},$$

according as $x$ is in the right or the left half plane; $M_i(x)$ is bounded for large values of $x$. The first form can also

be used in the left half plane, except near the axis of reals, since as $x \to \infty$ along any ray through the origin $|v|$ is proportional to $|x|$, and the factor of proportionality can be included in $M_i(x)$. Since $H_n(x) = T(x)\, H_n'(x)$,

$$
\begin{aligned}
h_{i2}(x) &= t_{i1}(x)\, h_{12}'(x) + t_{i2}(x)\, h_{22}'(x) \\
&= t_{i2}(x) + t_{i2}(x)\frac{M_2(x)}{x^{\lambda-1}} + t_{i1}(x)\left(\frac{\varrho_2}{\varrho_1}\right)^{x} \frac{M_1(x)}{x^{\lambda-1}}
\end{aligned}
$$

$$
(119)\qquad \left\{
\begin{aligned}
&= \varrho_2^{x}\, x^{-\beta_2-1}\Big[(s_{i2}+\cdots) + \frac{M_2(x)}{x^{\lambda-1}}(s_{i2}+\cdots) \\
&\qquad\qquad + x^{\beta_2-\beta_1}\frac{M_1(x)}{x^{\lambda-1}}(s_{i1}+\cdots)\Big].
\end{aligned}
\right.
$$

The second and third terms in the bracket can be made arbitrarily small by taking $k$ and hence $\lambda$ sufficiently large. Hence $h_{12}(x)$ and $h_{22}(x)$ are represented asymptotically by $s_{12}(x)$ and $s_{22}(x)$ respectively in the sector $-\pi < \arg x \leq \pi$. If $x \to \infty$ along a ray parallel to the negative axis of reals, $x^{\lambda-1}$ is replaced by $v^{\lambda-1}$ in (119), and we see that $h_{i2}(x)$ differs from $t_{i2}(x)$ by terms of the order of $v^{1-\lambda}$; hence the first $k$ terms of $s_{i2}(x)$ furnish a close approximation to $h_{i2}(x)$ if $|v|$ is taken sufficiently large.*

Consider the first order equation

$$
(120)\quad y(x+1) = |R(x)|\, y(x) = \frac{\varrho_1 \varrho_2 x}{x+\beta_1+\beta_2+2}\, y(x),
$$

where $|R(x)|$ is the determinant of the matrix $R(x)$. This is satisfied formally by the series (110), and by § 1, Chap. II, an analytic solution which is represented asymptotically by this series in the sector $-\pi < \arg x \leq \pi$ is furnished by the limit as $n \to \infty$ of the expression

$$
\frac{1}{|R(x)|}\,\frac{1}{|R(x+1)|}\cdots\frac{1}{|R(x+n)|}\,|T(x+n+1)|;
$$

* The behavior of $h_{12}(x)$ and $h_{22}(x)$ as $x \to \infty$ along such a ray is investigated further in § 9.

but this is simply the determinant of $H_n(x)$. Hence the latter converges uniformly to a limit function $D(x)$ which is analytic except for poles at $x = 0, -1, -2, \cdots$, where $|R(x)|, |R(x+1)|, \cdots$, vanish, and which is represented asymptotically by $|S(x)|$ in the sector $-\pi < \arg x < \pi$. Its explicit form, as we see from § 3, A, Chap. II, is

$$D(x) = (\varrho_2 - \varrho_1) s_1 s_2 \frac{\varrho_1^x \varrho_2^x \Gamma(x)}{\Gamma(x+\beta_1+\beta_2+2)}.$$

Since the asymptotic form of $\Gamma(x)$ holds to an arbitrary degree of approximation along a ray parallel to the negative axis of reals and sufficiently far above or below it, the same is true for $D(x)$.

In the same manner, by starting with the product $G_n(x)$ in (114), we can prove the second part of our existence theorem. The limit functions $g_{11}(x)$, $g_{21}(x)$, and $\overline{D}(x)$ will in general have poles at the points $-1-\beta_1-\beta_2, -\beta_1-\beta_2, 1-\beta_1-\beta_2, \cdots$, since elements of $R(x-1)$, $R(x-2), \cdots$, have poles at these points.* We have

$$\overline{D}(x) = (\varrho_2 - \varrho_1) s_1 s_2 \frac{\varrho_1^x \varrho_2^x \overline{\Gamma}(x)}{\overline{\Gamma}(x+\beta_1+\beta_2+2)}.$$

In proving the convergence of the elements of the second column of $H_n(x)$ and of the first column of $G_n(x)$, we make use of the fact that $|\varrho_1| \geq |\varrho_2|$. In dealing with the other column the rôles of $\varrho_1$ and $\varrho_2$ are interchanged, so their elements in general diverge. If, however, $|\varrho_1| = |\varrho_2|$, the elements of both columns converge, and give us two pairs of solutions which are represented asymptotically by the series $s_{11}(x)$, $s_{21}(x)$ and $s_{12}(x)$, $s_{22}(x)$, one pair in the sector $-\pi < \arg x < \pi$ and the other in the sector $0 < \arg x < 2\pi$.

The gap caused by the divergence of one column of $H_n(x)$ and $G_n(x)$ may be filled as follows. Let us define two functions $y_{11}(x)$, $y_{21}(x)$ by the equations

---

* Since the elements of the first row of $R(x-1)$ are constants, $g_{11}(x)$ does not have a pole at $-1-\beta_1-\beta_2$.

$$\begin{vmatrix} y_{11}(x) & h_{12}(x) \\ y_{21}(x) & h_{22}(x) \end{vmatrix} = \begin{vmatrix} y_{11}(x) & h_{12}(x) \\ y_{11}(x+1) & h_{12}(x+1) \end{vmatrix} = D(x);$$

the latter is a non-homogeneous difference equation in $y_{11}(x)$, and may be written

$$(121) \quad h_{12}(x)\,y_{11}(x+1) - h_{12}(x+1)\,y_{11}(x) = -D(x).$$

The associated homogeneous equation has the solution $h_{12}(x)$, and by § 5, Chap. I, eq. (121) has the solution

$$(122) \quad y_{11}(x) \doteq h_{12}(x)\,c(x) = h_{12}(x) \mathop{\Large\mathcal{S}} \frac{-D(x)}{h_{12}(x)\,h_{22}(x)}.$$

This and the function

$$\begin{aligned} y_{21}(x) &= y_{11}(x+1) = h_{12}(x+1)\,[c(x) + \Delta c(x)] \\ (123) \qquad &= h_{22}(x) \mathop{\Large\mathcal{S}} \frac{-D(x)}{h_{12}(x)\,h_{22}(x)} - \frac{D(x)}{h_{12}(x)} \end{aligned}$$

form a solution of the system (107); for since $h_{12}(x)$ and $h_{22}(x)$ satisfy (107) and $D(x)$ eq. (120), we have

$$\begin{aligned} y_{21}(x+1) &= h_{22}(x+1)\,[c(x) + \Delta c(x)] - \frac{D(x+1)}{h_{12}(x+1)} \\ = \Big[ &- \frac{\varrho_1 \varrho_2 x}{x + \beta_1 + \beta_2 + 2} h_{12}(x) \\ &+ \frac{(\varrho_1 + \varrho_2)(x+1) + \beta_1 \varrho_2 + \beta_2 \varrho_1}{x + \beta_1 + \beta_2 + 2} h_{22}(x) \Big] \Big[ c(x) - \frac{D(x)}{h_{12}(x)\,h_{22}(x)} \Big] \\ &- \frac{\varrho_1 \varrho_2 x}{x + \beta_1 + \beta_2 + 2} \frac{D(x)}{h_{22}(x)} \\ &= -\frac{\varrho_1 \varrho_2 x}{x + \beta_1 + \beta_2 + 2} y_{11}(x) + \frac{(\varrho_1 + \varrho_2)(x+1) + \beta_1 \varrho_2 + \beta_2 \varrho_1}{x + \beta_1 + \beta_2 + 2} y_{21}(x). \end{aligned}$$

Similarly, from the equations

$$\begin{vmatrix} g_{11}(x) & y_{12}(x) \\ g_{21}(x) & y_{22}(x) \end{vmatrix} = \begin{vmatrix} g_{11}(x) & y_{12}(x) \\ g_{11}(x+1) & y_{12}(x+1) \end{vmatrix} = D(x)$$

we obtain another solution of (107), namely

$$\begin{cases} y_{12}(x) = g_{11}(x) \mathop{\mathsf{S}} \dfrac{\bar{D}(x)}{g_{11}(x)\, g_{21}(x)}, \\ y_{22}(x) = g_{21}(x) \mathop{\mathsf{S}} \dfrac{\bar{D}(x)}{g_{11}(x)\, g_{21}(x)} + \dfrac{\bar{D}(x)}{g_{11}(x)}. \end{cases}$$

By suitably evaluating these sums we can obtain two sets of analytic solutions, as we shall see in the next two sections.

## § 5. The intermediate solutions.

THEOREM. *There exist two pairs of solutions* $y'_{11}(x)$, $y'_{21}(x)$; $y'_{12}(x)$, $y'_{22}(x)$ *and* $\bar{y}'_{11}(x)$, $\bar{y}'_{21}(x)$; $y'_{12}(x)$, $\bar{y}'_{22}(x)$ *analytic everywhere except within a limited distance of the axis of reals, and having the property that for large values of* $x$ $y'_{ij}(x) \sim s_{ij}(x)$ *in the sector* $-\pi < \arg x < \pi$ *and* $\bar{y}'_{ij}(x) \sim s_{ij}(x)$ *in the sector* $0 < \arg x < 2\pi$.

We may take $y'_{12}(x) = h_{12}(x)$, $y'_{22}(x) = h_{22}(x)$, since $h_{12}(x)$ and $h_{22}(x)$ have the properties stated. For the other solution of the first pair we use (122) and (123), evaluating the sum as an infinite series [cf. eq. (10)]:

$$(124) \qquad y'_{11}(x) = -h_{12}(x) \sum_{n=1}^{\infty} \frac{D(x-n)}{h_{12}(x-n)\, h_{22}(x-n)},$$

which represents an analytic function if it converges uniformly. To prove that this is the case when $x$ is sufficiently far from the axis of reals, consider the approximate form of a term of the series for such values of $x$, as determined in § 4. We have to terms of the $k$th order

$$\frac{D(x)}{h_{12}(x)\, h_{22}(x)}$$

$$\frac{(\varrho_1 \varrho_2)^x x^{-\beta_1-\beta_2-2}\left(d + \dfrac{d'}{x} + \cdots + \dfrac{d^{(k-1)}}{x^{k-1}}\right)}{\varrho_2^x x^{-\beta_2-1}\left(s_{12} + \cdots + \dfrac{s_{12}^{(k-1)}}{x^{k-1}}\right) \varrho_2^x x^{-\beta_2-1}\left(s_{22} + \cdots + \dfrac{s_{22}^{(k-1)}}{x^{k-1}}\right)}$$

$$(125) \qquad = \left(\frac{\varrho_1}{\varrho_2}\right)^x x^{\beta_2-\beta_1}\left(\eta + \frac{\eta'}{x} + \cdots + \frac{\eta^{(k-1)}}{x^{k-1}}\right);$$

where the $\eta$'s are known constants $\left(\text{in particular, } \eta = \frac{\varrho_2 - \varrho_1}{\varrho_2} \frac{s_1}{s_2}\right)$. The series in (124) may therefore be written

$$(126)\quad \sum_{n=1}^{\infty} \left(\frac{\varrho_1}{\varrho_2}\right)^{x-n} (x-n)^{\beta_2-\beta_1} \times \left[\eta + \frac{\eta'}{x-n} + \cdots + \frac{\eta^{(k-1)}}{(x-n)^{k-1}} + \frac{M(x-n)}{(x-n)^k}\right],$$

where $M(x)$ is a function which is bounded for large values of $|v|$. Let $|\varrho_1| > |\varrho_2|$; then the terms diminish in approximately geometrical ratio for large values of $n$, so the series converges uniformly in the neighborhood of points $x$ sufficiently far above or below the axis of reals. Hence $y'_{11}(x)$ as given by (124) and $y'_{21}(x) = y'_{11}(x+1)$ form a solution of (107) which is analytic above and below a certain horizontal strip enclosing the axis of reals.

In order to determine the asymptotic form of this solution, give $x$ a fixed value, and separate the series (126) into two parts: (1) the first $m$ terms in which $n < \frac{1}{2}|x|$, and (2) the remaining terms (cf. § 7, Chap. I). The first part may be written

$$\left(\frac{\varrho_1}{\varrho_2}\right)^x x^{\beta_2-\beta_1} \sum_{n=1}^{m} \left(\frac{\varrho_2}{\varrho_1}\right)^n \left[1 - \frac{(\beta_2-\beta_1)n}{x} + \cdots + M_0(x)\left(\frac{n}{x}\right)^k\right]$$
$$\times \left\{\eta + \frac{\eta'}{x}\left[1 + \frac{n}{x} + \cdots + M_1(x)\left(\frac{n}{x}\right)^{k-1}\right] + \cdots\right\}$$
$$= \left(\frac{\varrho_1}{\varrho_2}\right)^x x^{\beta_2-\beta_1} \sum_{n=1}^{m} \left(\frac{\varrho_2}{\varrho_1}\right)^n \times \left[\eta + \frac{\eta' - \eta(\beta_2-\beta_1)n}{x} + \cdots + M'(x)\left(\frac{n}{x}\right)^k\right],$$

where the functions $M_i(x)$ and $M'(x)$ are bounded, since the error caused by breaking off the convergent binomial expansions at the $k$th term is of the order of $(n/x)^k$. The typical expression to be examined is

$$\sum_{n=1}^{m} \left(\frac{\varrho_2}{\varrho_1}\right)^n \frac{c\,n^\lambda}{x^\mu} = \frac{c}{x^\mu} \sum_{n=1}^{m} \left(\frac{\varrho_2}{\varrho_1}\right)^n n^\lambda,$$

where $c$ is a constant and $\lambda$ and $\mu$ are integers from the set $0, 1, 2, \cdots, k$. A comparison of the sum on the right hand side with the sums $\sum j^\lambda r^j$ evaluated in § 7, Chap. I, shows that the former is represented asymptotically by a constant for every value of $\lambda$ (for $\lambda = \mu = 0$, the constant is $\varrho_2/(\varrho_1 - \varrho_2)$). The sum which involves $M'(x)$ is of the order of $x^{-k}$, since

$$\sum_{n=1}^{m} \left(\frac{\varrho_2}{\varrho_1}\right)^n M'(x) \left(\frac{n}{x}\right)^k < \frac{M}{x^k} \sum_{n=1}^{\infty} \left(\frac{\varrho_2}{\varrho_1}\right)^n n^k = \frac{MN}{x^k}.$$

where $M$ and $N$ are constants. The first $m$ terms of the series (126) are therefore represented asymptotically by

$$\left(\frac{\varrho_1}{\varrho_2}\right)^x x^{\beta_2 - \beta_1}\left(a + \frac{a'}{x} + \frac{a''}{x^2} + \cdots\right),$$

where the $a$'s are constants (in particular, $a = -s_1/s_2$). The terms after the $m$th diminish in approximately geometrical ratio, and they are all multiplied by the factor $(\varrho_2/\varrho_1)^m$, which approaches zero exponentially as $x$ increases; hence the second part of the series does not affect the asymptotic form.

Substituting this expression in (124), and using the known asymptotic form of $h_{12}(x)$, which holds in the sector $-\pi < \arg x < \pi$, we have

$$\begin{aligned} y'_{11}(x) &\sim -\varrho_2^x x^{-\beta_2 - 1}\left(s_2 + \frac{s'_{12}}{x} + \cdots\right) \\ &\qquad\qquad \times \left(\frac{\varrho_1}{\varrho_2}\right)^x x^{\beta_2 - \beta_1}\left(-\frac{s_1}{s_2} + \frac{a'}{x} + \cdots\right) \\ &\sim \varrho_1^x x^{-\beta_1 - 1}\left(s_1 + \frac{b'}{x} + \frac{b''}{x^2} + \cdots\right) \end{aligned}$$

in this sector. For the other element of the solution we have

$$y'_{21}(x) = y'_{11}(x+1) \sim \varrho_1^x x^{-\beta_1 - 1}\left(\varrho_1 s_1 + \frac{c'}{x} + \frac{c''}{x^2} + \cdots\right).$$

Since these asymptotic expressions are formally equal to $y'_{11}(x)$ and $y'_{21}(x)$, they must satisfy formally the system (107); they are series of the same form as $s_{11}(x)$ and $s_{21}(x)$, and

they have precisely the same leading terms; hence they must be identical with the latter, on account of the uniqueness of these series. Thus $y'_{11}(x) \sim s_{11}(x)$ and $y'_{21}(x) \sim s_{21}(x)$ in the sector $-\pi < \arg x < \pi$.

In the argument above we took $|\varrho_1| > |\varrho_2|$; if $|\varrho_1| = |\varrho_2|$, as remarked in § 4, both columns of $H_n(x)$ converge, and the two solutions thus obtained satisfy the conditions of our theorem.

Similarly we can prove the existence of the second pair of intermediate solutions, namely $\bar{y}'_{11}(x) = g_{11}(x)$, $y'_{21}(x) = g_{21}(x)$ and

$$\begin{cases} y'_{12}(x) = -g_{11}(x)\sum\limits_{n=0}^{\infty}\dfrac{D(x+n)}{g_{11}(x+n)\,g_{21}(x+n)}, \\ \bar{y}'_{22}(x) = y'_{12}(x+1). \end{cases}$$

The two solutions in each pair are linearly independent, since one is represented asymptotically by $s_{11}(x)$, $s_{21}(x)$ and the other by $s_{12}(x)$, $s_{22}(x)$. The two pairs of intermediate solutions may therefore be combined into two matrix solutions $Y'(x)$ and $Y'(x)$ of eq. (108). It is evident from the definitions of the solutions $y_{11}(x)$ $y_{21}(x)$ and $y_{12}(x)$, $y_{22}(x)$ at the end of § 4 that the determinants of these matrices are equal to $D(x)$ and $D(x)$ respectively.

The intermediate solutions are discussed further at the end of § 8.

## § 6. The principal solutions.

By using a suitable contour integral to evaluate the sum in (122) we obtain an important set of solutions called the *principal solutions.*

THEOREM. *There exist two solutions $h_{11}(x)$, $h_{21}(x)$ and $h_{12}(x)$, $h_{22}(x)$ analytic throughout the finite part of the plane except for poles, and such that $h_{ij}(x) \sim s_{ij}(x)$ in any right half plane.* There exist two other solutions $g_{11}(x)$, $g_{21}(x)$ and $g_{12}(x)$, $g_{22}(x)$ analytic except for poles, and such that $g_{ij}(x) \sim s_{ij}(x)$ in any left half plane. These properties in general determine the solutions uniquely.*

* By "any right (or left) half plane" is meant the portion of the plane to the right (or left) of an arbitrary line parallel to the axis of imaginaries.

We can obviously take for $h_{12}(x)$, $h_{22}(x)$ and $g_{11}(x)$, $g_{21}(x)$ the solutions already denoted by these symbols, since they have the required properties. In case $|\varrho_1| = |\varrho_2|$, the other two solutions furnished by the matrices $H_n(x)$ and $G_n(x)$ may be taken as $h_{11}(x)$, $h_{21}(x)$ and $g_{12}(x)$, $g_{22}(x)$ respectively. Accordingly in what follows we will assume that $|\varrho_1| > |\varrho_2|$.

Let

$$\Phi(x) = \frac{-D(x)}{h_{12}(x)\,h_{22}(x)}; \tag{127}$$

we see from the approximate form (125) that $\Phi(x)$ is analytic everywhere except within a limited distance of the negative axis of reals. Consider the integral

$$I(x) = \int_L e^{2\lambda\pi i(x-t)} \frac{\Phi(t)\,dt}{1-e^{2\pi i(x-t)}}\,.$$

where $\lambda$ is an integer and $L$ is the contour $\infty AB\infty$ of Fig. 6, consisting of two straight lines of inclination $\varepsilon$ and $\pi-\varepsilon$, where $\varepsilon$ is a small positive angle, one below and one

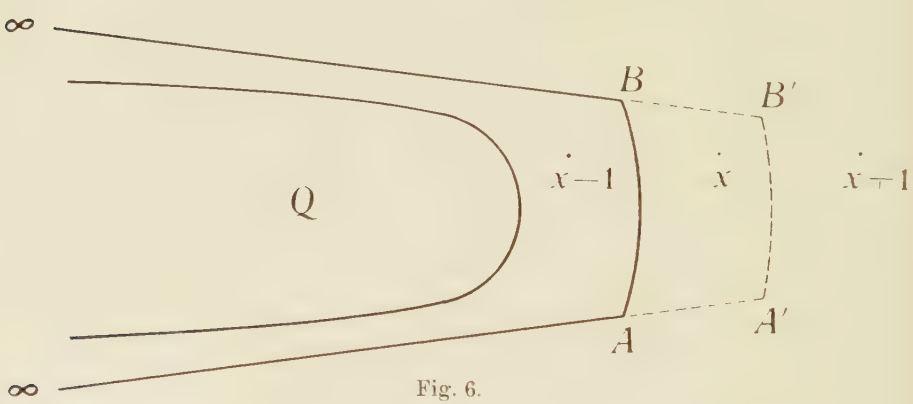


Fig. 6.

above the region $Q$ in which $\Phi(t)$ is not everywhere analytic, and a simple curve passing between $t = x$ and $t = x-1$, $x$ being a point in the right half plane far enough to the right so that the region $Q$ lies wholly within the contour.*

* The exact positions of the lines and the curve are of course immaterial.

Using (125), we see that the behavior of the integrand as $t \to \infty$ along the lines $A\infty$ and $B\infty$ depends on the exponential factors $e^{2(1-\lambda)\pi i t}(\varrho_1/\varrho_2)^t$, or, if we use only the exponents, on

$$[2(1-\lambda)\pi i + \log \varrho_1 - \log \varrho_2]\, t.$$

If we write $t = \sigma + i\tau$, the real part of this is

$$\sigma(\log|\varrho_1| - \log|\varrho_2|) - 2\pi\tau\left(1 - \lambda + \frac{\arg \varrho_1 - \arg \varrho_2}{2\pi}\right);$$

since $\sigma < 0$ in the second and third quadrants, and since $|\varrho_1| > |\varrho_2|$, this is negative if $|\tau/\sigma|$ is sufficiently small. Hence if the angle $\varepsilon$ is taken sufficiently small, the integrand approaches zero exponentially along these lines; the integral therefore converges uniformly in the neighborhood of the point $x$. To show that $I(x)$ gives us the sum of $\Phi(x)$, form the difference $\Delta I(x)$; this is equal to the integral of the same integrand over the closed path $AA'B'BA$; but the integrand is analytic inside this region except for a simple pole at $t = x$, where its residue is $\Phi(x)/2\pi i$; hence $\Delta I(x) = \Phi(x)$, or $I(x) = \mathfrak{S}\, \Phi(x)$.

Using this integral in (122), we obtain the solution

$$(128) \qquad \begin{cases} h_{11}(x) = h_{12}(x) \displaystyle\int_L e^{2\lambda\pi i(x-t)} \frac{\Phi(t)\, dt}{1 - e^{2\pi i(x-t)}}, \\ h_{21}(x) = h_{11}(x+1), \end{cases}$$

which is analytic in a certain right half plane. But by repeated use of eqs. (107) or (108) we can express $h_{11}(x)$ and $h_{21}(x)$ in terms of $h_{11}(x+m)$ and $h_{21}(x+m)$, where $m$ is a positive integer which may be taken so large that the solution is analytic at $x+m$ and hence at $x$ itself. This is true even if $x$ is near the negative axis of reals; the only exceptions are the points $x = 0, -1, -2, \cdots$, where the determinants $|R(x)|$, $|R(x+1)|, \cdots$, vanish; at those points the solution has poles.

It remains to consider the asymptotic form of the solution. For this purpose we will separate the contour $L$ into two

parts as follows: let $L_1$ be a fixed contour $\infty A_1 B_1 \infty$ consisting of two straight lines of inclination $\varepsilon$ and $\pi-\varepsilon$ and a simple curve, enclosing all the singularities of $\Phi(t)$; let $L_2$ be a loop enclosing all of the points $x-1, x-2, \cdots$ which lie to the right of $L_1$, but not $x, x+1, \cdots$. These two contours together are obviously equivalent to $L$.

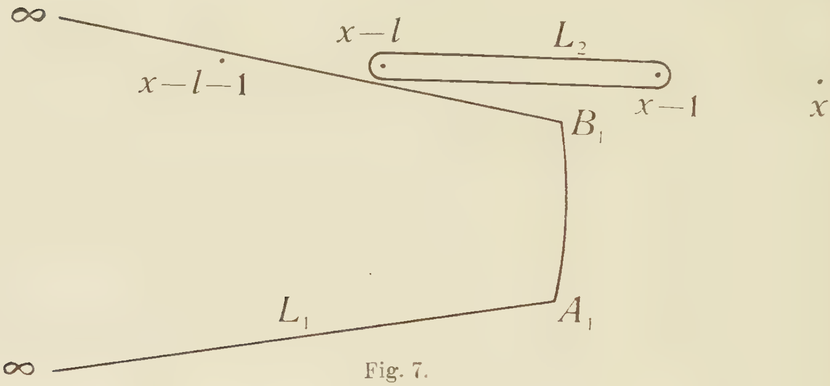


Fig. 7.

The part of the integral contributed by $L_1$ is a periodic function of $x$, since the integrand is a periodic function of $x$ and the contour is fixed. It may be written in either of the following two ways:

$$e^{2\lambda\pi i x}\int_{L_1} e^{-2\lambda\pi i t}\,\frac{\Phi(t)\,dt}{1-e^{2\pi i(x-t)}}\,.$$

$$e^{2(\lambda-1)\pi i x}\int_{L_1} e^{-2(\lambda-1)\pi i t}\,\frac{\Phi(t)\,dt}{e^{-2\pi i(x-t)}-1}\,;$$

in the first form the denominator is bounded as $v\to+\infty$, since $e^{2\pi i x}$ then approaches zero; in the second form the denominator is bounded as $v\to-\infty$. Hence the part of $h_{11}(x)$ contributed by $L_1$ may be written in the form

$$h_{12}(x)\,e^{2\lambda\pi i x}\,q(x) \quad \text{or} \quad h_{12}(x)\,e^{2(\lambda-1)\pi i x}\,q'(x),$$

according as $x$ is in the upper or lower half plane; $q(x)$ and $q'(x)$ denote periodic functions which are bounded as $v\to\pm\infty$;

and $v \to -\infty$ respectively. Either form may be used if $x$ is on the axis of reals.

Consider now the part contributed by $L_2$. Since the integrand is analytic within $L_2$ except for simple poles at $t = x-1, x-2, \cdots, x-l$, the integral can be expressed in terms of the residues at these points; we have namely

$$h_{12}(x) \sum_{n=1}^{l} \Phi(x-n).$$

which consists of the first $l$ terms of $y'_{11}(x)$ [eq. (124)]; let us call it $y''_{11}(x)$. Thus we have

$$h_{11}(x) = \begin{cases} h_{12}(x)\, e^{2\lambda\pi i x}\, q(x) + y''_{11}(x), \\ h_{12}(x)\, e^{2(\lambda-1)\pi i x}\, q'(x) + y''_{11}(x), \end{cases}$$

according as $x$ is in the upper or lower half plane.

The asymptotic form of $y''_{11}(x)$ in the right half plane is the same as that of $y'_{11}(x)$, since, as we saw in § 5, the asymptotic form of the latter is determined by the first $m$ terms of the series; the argument in § 5 applies without change even when $x$ is near the axis of reals. Hence we have in the first quadrant (including the axis of reals)

$$h_{11}(x) \sim s_{12}(x)\, e^{2\lambda\pi i x}\, q(x) + s_{11}(x), \tag{129}$$

and we need to determine which of the two terms has the greater order of magnitude as $x \to \infty$. This depends on the exponential factors $\varrho_2^x e^{2\lambda\pi i x}$, $\varrho_1^x$ or, if we divide by the second one and use only the exponents, on

$$2\pi i\left(\lambda + \frac{\log \varrho_2 - \log \varrho_1}{2\pi i}\right) x, \;\; 0.$$

So far we have put no conditions on the integer $\lambda$, and hence have not completely specified the solution $h_{11}(x)$ which we are considering. Let us define $\lambda$ as the smallest integer which exceeds the real part of

$$\frac{\log \varrho_1 - \log \varrho_2}{2\pi i}; \tag{130}$$

that is, let $\lambda$ be the integer such that

$$\lambda - 1 \leqq \frac{\arg \varrho_1 - \arg \varrho_2}{2\pi} < \lambda.$$

By choosing suitable determinations of $\arg \varrho_1$ and $\arg \varrho_2$ we may always have

$$\arg \varrho_1 - \pi < \arg \varrho_2 \leqq \arg \varrho_1 + \pi; \tag{131}$$

then $\lambda = 1$ if $\arg \varrho_1 \geqq \arg \varrho_2$ and $\lambda = 0$ if $\arg \varrho_1 < \arg \varrho_2$. We shall limit ourselves in general to such determinations.

With this definition of $\lambda$, the real part of the expression $\lambda + (\log \varrho_2 - \log \varrho_1)/2\pi i$ lies between 0 and 1; the imaginary part is $(\log|\varrho_2| - \log|\varrho_1|)/2\pi i$, where the numerator is negative, since $|\varrho_1| > |\varrho_2|$. Hence we can write

$$\lambda + \frac{\log \varrho_2 - \log \varrho_1}{2\pi i} = a - \frac{b}{2\pi i}, \tag{132}$$

where $0 < a \leqq 1$ and $b > 0$. Since $x$ is in the first quadrant, we have $u > 0$, $v \geqq 0$. The real part of the first exponent above is then

$$-bu - 2\pi a v, \tag{133}$$

which becomes negatively infinite as $x \to \infty$; the second exponent has therefore the greater real part, so the second term in (129) is the dominant one, and $h_{11}(x) \sim s_{11}(x)$ in the first quadrant.

In the fourth quadrant we have

$$h_{11}(x) \sim s_{12}(x)\, e^{2(\lambda-1)\pi i x}\, q'(x) + s_{11}(x); \tag{134}$$

here the dominant term depends on the exponents

$$2\pi i \left(\lambda - 1 + \frac{\log \varrho_2 - \log \varrho_1}{2\pi i}\right) x, \quad 0.$$

The real part of the first is

$$-bu + 2\pi(1-a)v, \tag{135}$$

which becomes negatively infinite, since $v < 0$, so the second term dominates again and $h_{11}(x) \sim s_{11}(x)$ also in the fourth quadrant. The only exception is when $a = 1$, in which case the asymptotic form does not hold along rays parallel to the axis of imaginaries, since (135) then remains finite, so that the two terms in (129) are of equal order of magnitude. If however we use in place of the right half plane a half plane bounded by a line inclined slightly to the vertical, i. e., a sector $-\pi/2 + \varepsilon \leq \arg x \leq \pi/2 + \varepsilon$, where $\varepsilon$ is a small positive angle, $h_{11}(x)$ has the same asymptotic properties as when $a < 1$; for the relation (129) holds throughout the upper half plane, and (133) remains negative in that part of the second quadrant for which $v/(-u) > b/2\pi a$, so that $h_{11}(x) \sim s_{11}(x)$ in the above sector if $\varepsilon < \tan^{-1}(2\pi a/b)$, regardless of the value of $a$.

If instead of *the* right half plane $u > 0$ we consider *any* right half plane $u > c$, where $c$ is any real constant, the first term in (133) and (135) may be positive, but it remains finite, so for large values of $|v|$ these expressions are negative and our conclusions hold as before.

The other element $h_{21}(x)$ of the solution is equal to $h_{11}(x+1)$, so it is represented asymptotically in any right half plane by $s_{11}(x+1) = s_{21}(x)$. It is evident from their asymptotic forms that the solutions $h_{11}(x)$, $h_{21}(x)$ and $h_{12}(x)$, $h_{22}(x)$ are linearly independent; hence they may be combined into a matrix solution $H(x)$ of eq. (108). The determinant of this matrix is equal to $D(x)$ (cf. end of § 5).

For the remaining solution we write

$$(136) \qquad \begin{cases} g_{12}(x) = g_{11}(x) \displaystyle\int_{L'} e^{2\lambda'\pi i(x-t)} \frac{\overline{\Phi}(t)\, dt}{e^{2\pi i(x-t)} - 1}, \\ g_{22}(x) = g_{12}(x+1), \end{cases}$$

where

$$\overline{\Phi}(t) = \frac{\overline{D}(t)}{g_{11}(t)\, g_{21}(t)}$$

and $\lambda'$ is the smallest integer which exceeds or equals the real part of $(\log \varrho_2 - \log \varrho_1)/2\pi i$ (i. e., $\lambda' = 1 - \lambda$); the con-

tour $L'$ is similar to $L$, but extends to the right instead of to the left. A discussion similar to that above shows that this solution has the properties stated in the theorem. It may be combined with $g_{11}(x)$, $g_{21}(x)$ to form a matrix solution $G(x)$, whose determinant is equal to $D(x)$.

Since the first element of any solution of the system (107) is a solution of the single equation (103) of the second order, we have proved that the latter equation possesses four solutions $h_1(x)$, $h_2(x)$, $g_1(x)$, $g_2(x)$ [$= h_{11}(x)$, $h_{12}(x)$, $g_{11}(x)$, $g_{12}(x)$ respectively] such that $h_1(x) \sim S_1(x)$ and $h_2(x) \sim S_2(x)$ in any right half plane, and $g_1(x) \sim S_1(x)$ and $g_2(x) \sim S_2(x)$ in any left half plane. The former two are called the *first principal system* or the *first canonical system* of solutions of eq. (103), and the latter two the *second principal system* or *second canonical system* of solutions.

It remains to prove that there are no other solutions possessing the same properties. Consider first the solution $h_{12}(x)$, $h_{22}(x)$; assume that there exists another solution $h'_{12}(x)$, $h'_{22}(x)$ different from this such that $h'_{12}(x) \sim s_{12}(x)$ and $h'_{22}(x) \sim s_{22}(x)$ in any right half plane. Since the solutions $h_{11}(x)$, $h_{21}(x)$ and $h_{12}(x)$, $h_{22}(x)$ are linearly independent, we can write

$$(137) \qquad \begin{cases} h'_{12}(x) = p_1(x)\, h_{11}(x) + p_2(x)\, h_{12}(x), \\ h'_{22}(x) = p_1(x)\, h_{21}(x) + p_2(x)\, h_{22}(x). \end{cases}$$

where $p_1(x)$ and $p_2(x)$ are periodic functions. If we divide the first equation by $h'_{12}(x)$ and then let $x \to \infty$ parallel to the positive axis of reals, we have

$$1 \sim p_1(x) \frac{s_{11}(x)}{s_{12}(x)} + p_2(x),$$

or

$$1 \sim p_1(x) \left( \frac{\varrho_1}{\varrho_2} \right)^x x^{\beta_2 - \beta_1} \left( \frac{s_1}{s_2} + \cdots \right) + p_2(x).$$

If $|\varrho_1| > |\varrho_2|$, the factor $(\varrho_1/\varrho_2)^x$ becomes infinite as $x \to \infty$; hence $p_1(x)$ must approach zero; but since $p_1(x)$ is periodic, this means that it vanishes identically. It follows that $p_2(x) \sim 1$, and since $p_2(x)$ is also periodic, $p_2(x) = 1$ identically.

Hence in this case $h'_{12}(x) = h_{12}(x)$, contrary to hypothesis. If $|\varrho_1| = |\varrho_2|$, the same argument applies if $R(\beta_2) > R(\beta_1)$. But both $h_{12}(x)$ and $h'_{12}(x)$ are single-valued analytic functions of $\beta_1$ and $\beta_2$ for large values of $x$ in the right half plane; hence the identity $h'_{12}(x) = h_{12}(x)$ must hold even when $R(\beta_2) \leqq R(\beta_1)$. Hence $h_{12}(x)$, $h_{22}(x)$ is the only solution of (107) which is represented asymptotically by $s_{12}(x)$, $s_{22}(x)$ in any right half plane. A similar argument shows that $g_{11}(x)$, $g_{21}(x)$ is the only solution which is represented asymptotically by $s_{11}(x)$, $s_{21}(x)$ in any left half plane.

Consider now the solution $h_{11}(x)$, $h_{21}(x)$. Assuming that there exists a different solution $h'_{11}(x)$, $h'_{21}(x)$ represented asymptotically by $s_{11}(x)$, $s_{21}(x)$ in any right half plane, we can express the latter in terms of the $h$'s as in (137); solving for $p_1(x)$, we have

$$p_1(x) = \frac{\begin{vmatrix} h'_{11}(x) & h_{12}(x) \\ h'_{21}(x) & h_{22}(x) \end{vmatrix}}{\begin{vmatrix} h_{11}(x) & h_{12}(x) \\ h_{21}(x) & h_{22}(x) \end{vmatrix}}.$$

If $x \to \infty$ parallel to the positive axis of reals, both numerator and denominator are represented asymptotically by $|S(x)|$, so $p_1(x) \sim 1$, i. e., $p_1(x) = 1$. Using this value, we see that

$$p_2(x) = \frac{h'_{11}(x) - h_{11}(x)}{h_{12}(x)};$$

this is single-valued and analytic in a period strip sufficiently far to the right, except possibly at the ends, where we see from the asymptotic forms that

$$\lim_{x \to \infty} p_2(x) \left(\frac{\varrho_2}{\varrho_1}\right)^x x^{\beta_1 - \beta_2} = 0.$$

Writing $p_2(x) = q(z)$, where $z = e^{2\pi i x}$, we have at the upper end of the strip

$$\lim_{z \to 0} q(z) z^{\frac{\log \varrho_2 - \log \varrho_1}{2\pi i}} (\log z)^{\beta_1 - \beta_2} = 0,$$

or, by (132),

$$\lim_{z\to 0} q(z)z^{-\lambda} z^{a-\frac{b}{2\pi i}} (\log z)^{\beta_1-\beta_2} = 0,$$

where $0 < a \leq 1$. If $a \neq 1$, we see that $q(z)z^{-\lambda}$ cannot have a pole at $z = 0$, since the other factors do not vanish to an order as high as the first; and since the same limit is approached as $z \to \infty$ if $a \neq 1$, $q(z)z^{-\lambda}$ must vanish at $z = \infty$. The single-valued analytic function $q(z)z^{-\lambda}$ therefore has no singularity in the entire $z$-plane, so it must be a constant, namely zero. It follows that $p_2(x) = 0$ and $h'_{11}(x) = h_{11}(x)$, which proves that the solution $h_{11}(x), h_{21}(x)$ is uniquely determined by its asymptotic properties. A similar argument shows that the solution $g_{12}(x)$, $g_{22}(x)$ is also unique if $a \neq 1$.

In the exceptional case $a = 1$, the arguments of $\varrho_1$ and $\varrho_2$ are the same (or differ by a multiple of $2\pi$), so the real part of the expression (130) is zero; if we replace the value $\lambda = 1$ by $\lambda = 0$ the solutions which we get have precisely the same asymptotic properties in the sector $-\pi/2 < \arg x < \pi/2$, since the expressions (133) and (135) remain negative if $a = 0$ instead of 1. There is thus a real indeterminateness in this case.

In the argument above it is not necessary to assume that the asymptotic form of $h'_{ij}(x)$ agrees with that of $s_{ij}(x)$ beyond the first term; hence a solution of eq. (103) is identified as one of the principal solutions if the first term of its asymptotic form coincides with the first term of $S_1(x)$ or $S_2(x)$ in the right or left half plane.

## § 7. Integral and series solutions.

So far we have proved the existence of several solutions of eq. (103), characterized by certain asymptotic properties. We will now obtain solutions in terms of definite integrals and convergent infinite series, among which can be identified the solutions already obtained. For this purpose we make use of the *Laplace transformation*

(138)
$$y(x) = \int_a^b t^{x-1} v(t)\, dt$$

[cf. eq. (68)], and seek to determine a function $v(t)$ and a path of integration which will make this integral a solution of eq. (103).

By integration by parts,

$$\int_a^b x\, t^{x+k-1} v(t)\, dt = [t^{x+k} v(t)]_a^b - \int_a^b t^x \frac{d}{dt}[t^k v(t)]\, dt.$$

Substituting the integral (138) in eq. (103), and using this formula for $k = 0$, 1, and 2, we find that the left hand side reduces to

$$\int_a^b t^x \Big\{ -\frac{d}{dt}[(t-\varrho_1)(t-\varrho_2)v(t)]$$
$$+ v(t)[(\beta_1+\beta_2+2)t - (\beta_1+1)\varrho_2 - (\beta_2+1)\varrho_1]\Big\}\, dt$$
$$+ [t^x(t-\varrho_1)(t-\varrho_2)v(t)]_a^b.$$

This will be equal to zero if the integrand vanishes identically and if $a$ and $b$ are so chosen that the integrated part also vanishes. Equating the integrand to zero and simplifying, we find that $v(t)$ must satisfy the differential equation

$$\frac{v'(t)}{v(t)} = \frac{\beta_1}{t-\varrho_1} + \frac{\beta_2}{t-\varrho_2},$$

a solution of which is

$$v(t) = (t-\varrho_1)^{\beta_1}(t-\varrho_2)^{\beta_2}.$$

Eq. (103) is therefore satisfied by the integral

(139)
$$y(x) = \int_a^b t^{x-1}(t-\varrho_1)^{\beta_1}(t-\varrho_2)^{\beta_2}\, dt,$$

provided $a$ and $b$ are so chosen that

(140)
$$[t^x(t-\varrho_1)^{\beta_1+1}(t-\varrho_2)^{\beta_2+1}]_a^b = 0.$$

The expression in brackets vanishes at $t = 0$ if $R(x) > 0$; at $t = \varrho_1$ if $R(\beta_1) > -1$; at $t = \varrho_2$ if $R(\beta_2) > -1$; and

at $t=\infty$ if $R(x+\beta_1+\beta_2)<-2$. By taking for $a$ and $b$ any two of these four values, we obtain six solutions of eq. (103) of the form (139), each of which is valid under certain conditions on $x$, $\beta_1$, and $\beta_2$, namely

$$(141)\quad \begin{cases} h_1(x)=\int_0^{\varrho_1} t^{x-1}(t-\varrho_1)^{\beta_1}(t-\varrho_2)^{\beta_2}\,dt, \\ \qquad\qquad [R(x)>0,\ R(\beta_1)>-1]; \\ h_2(x)=\int_0^{\varrho_2} t^{x-1}(t-\varrho_1)^{\beta_1}(t-\varrho_2)^{\beta_2}\,dt, \\ \qquad\qquad [R(x)>0,\ R(\beta_2)>-1]; \\ g_1(x)=\int_\infty^{\varrho_1} t^{x-1}(t-\varrho_1)^{\beta_1}(t-\varrho_2)^{\beta_2}\,dt, \\ \qquad\qquad [R(x+\beta_1+\beta_2)<0^*,\ R(\beta_1)>-1]; \\ g_2(x)=\int_\infty^{\varrho_2} t^{x-1}(t-\varrho_1)^{\beta_1}(t-\varrho_2)^{\beta_2}\,dt, \\ \qquad\qquad [R(x+\beta_1+\beta_2)<0^*,\ R(\beta_2)>-1]; \\ l(x)=\int_{\varrho_2}^{\varrho_1} t^{x-1}(t-\varrho_1)^{\beta_1}(t-\varrho_2)^{\beta_2}\,dt, \\ \qquad\qquad [R(\beta_1)>-1,\ R(\beta_2)>-1]; \\ m(x)=\int_0^{\infty} t^{x-1}(t-\varrho_1)^{\beta_1}(t-\varrho_2)^{\beta_2}\,dt, \\ \qquad\qquad [R(x)>0,\ R(x+\beta_1+\beta_2)<0^*]. \end{cases}$$

To make these solutions perfectly definite, it is necessary to specify the paths of integration and the determinations of the multiple-valued factors of the integrand which we are using. We will take the paths of integration in $h_1(x)$ and $h_2(x)$ as the straight lines joining 0 to $\varrho_1$ and $\varrho_2$, and in $g_1(x)$ and $g_2(x)$ as the prolongations of these lines; accordingly in $h_1(x)$ and $g_1(x)$ we naturally take $\arg t=\arg\varrho_1$, and in $h_2(x)$ and $g_2(x)$ $\arg t=\arg\varrho_2$. Similarly in $g_1(x)$ and $g_2(x)$ we take $\arg(t-\varrho_1)=\arg\varrho_1$ and $\arg(t-\varrho_2)=\arg\varrho_2$ respectively. In $h_1(x)$ and $h_2(x)$ we will take $\arg(t-\varrho_1)$

* The integrals $g_1(x)$, $g_2(x)$, and $m(x)$ converge absolutely and uniformly as $t\to\infty$ in the neighborhood of any point $x$ in the half plane $R(x+\beta_1+\beta_2)<0$; hence the integrals represent solutions of eq. (103) in this half plane, even though the condition (140) is satisfied only in the half-plane $R(x+\beta_1+\beta_2)<-2$. Cf. Osgood: *Lehrbuch der Funktiontheorie* I (2nd ed.), pp. 450–452.

$= \arg \varrho_1 + \pi$ and $\arg (t - \varrho_2) = \arg \varrho_2 + \pi$ respectively. The argument of the remaining factor in each integrand varies as we move along the path; if $\arg \varrho_1 \geqq \arg \varrho_2$, we will take $\arg (t - \varrho_2)$ between $\arg \varrho_2$ and $\arg \varrho_2 + \pi$ in $h_1(x)$ and $g_1(x)$, and $\arg (t - \varrho_1)$ between $\arg \varrho_1 + \pi$ and $\arg \varrho_1 + 2\pi$ in $h_2(x)$ and $g_2(x)$; if $\arg \varrho_1 < \arg \varrho_2$, we will take $\arg (t - \varrho_2)$ between $\arg \varrho_2 + \pi$ and $\arg \varrho_2 + 2\pi$ in $h_1(x)$ and $g_1(x)$, and $\arg (t - \varrho_1)$ between $\arg \varrho_1$ and $\arg \varrho_1 + \pi$ in $h_2(x)$ and $g_2(x)$. If $\arg \varrho_1 = \arg \varrho_2$, we will let the path of integration in $h_1(x)$ avoid the point $\varrho_2$ by passing along a small semicircular arc to the left of $\varrho_2$, and let the path in $g_2(x)$ avoid the point $\varrho_1$ in a similar fashion (cf. Fig. 10). In $l(x)$ we will take the path of integration as the straight line from $\varrho_2$ to $\varrho_1$, and let $\arg t$ go from $\arg \varrho_2$ to $\arg \varrho_1$. If $\arg \varrho_1 \geqq \arg \varrho_2$, we will take for $\arg (t - \varrho_1)$ the value which lies between $\arg \varrho_1 + \pi$ and $\arg \varrho_1 + 2\pi$, and for $\arg (t - \varrho_2)$ the value between $\arg \varrho_2$ and $\arg \varrho_2 + \pi$; if $\arg \varrho_1 < \arg \varrho_2$, we will take for $\arg (t - \varrho_1)$ the value between $\arg \varrho_1$ and $\arg \varrho_1 + \pi$, and for $\arg (t - \varrho_2)$ the value between $\arg \varrho_2 + \pi$ and $\arg \varrho_2 + 2\pi$. In $m(x)$ we will take for the path of integration any straight line from 0 to $\infty$ in the sector $\arg \varrho_1 - 2\pi < \arg t < \arg \varrho_2$ if $\arg \varrho_1 \geqq \arg \varrho_2$, or in the sector $\arg \varrho_2 - 2\pi < \arg t < \arg \varrho_1$ if $\arg \varrho_1 < \arg \varrho_2$; for definiteness let us take the ray which passes through the point $-\varrho_1$. On this ray we will take $\arg t = \arg \varrho_1 - \pi$ and $\arg (t - \varrho_1) = \arg \varrho_1 + \pi$; if $\arg \varrho_1 \geqq \arg \varrho_2$, we will take $\arg (t - \varrho_2)$ between $\arg \varrho_2 + \pi$ and $\arg \varrho_2 + 2\pi$, and if $\arg \varrho_1 < \arg \varrho_2$ we will take it between $\arg \varrho_2$ and $\arg \varrho_2 + \pi$.

The restrictions on $x$, $\beta_1$, and $\beta_2$ may be avoided by replacing the straight line paths by suitable contours; thus if $R(\beta_1) \leqq -1$, we may use for $h_1(x)$ a loop circuit $l_1$ starting and ending at $t = 0$ and passing around $t = \varrho_1$ in the positive direction, and for $g_1(x)$ a loop $L_1$ starting and ending at $t = \infty$ and passing around $t = \varrho_1$ in the positive direction (Fig. 8).* The argument of $t - \varrho_1$ may be taken as increasing

* Cf. Nörlund: *Differenzenrechnung*, Chap. XI.

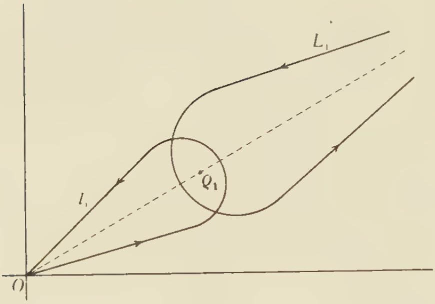


Fig. 8.

from $\arg \varrho_1 + \pi$ to $\arg \varrho_1 + 3\pi$ on $l_1$, and from $\arg \varrho_1$ to $\arg \varrho_1 + 2\pi$ on $L_1$, while $\arg t$ has values near $\arg \varrho_1$, and $\arg(t - \varrho_2)$ is determined in the same way as for the straight line integrals. The solutions $h_1'(x)$ and $g_1'(x)$ thus obtained are each valid in a half plane, and they differ from $h_1(x)$ and $g_1(x)$ as previously defined, in case the latter exist, only by a constant factor, namely

$$(142) \quad h_1'(x) = (1 - e^{2\pi i\beta_1})\, h_1(x), \quad g_1'(x) = (1 - e^{2\pi i\beta_1})\, g_1(x),$$

as we see by shrinking each loop down to a line segment and a small circle about $\varrho_1$.

All the restrictions are removed if we use double loop circuits, like that described in § 3, Chap. II (cf. Fig. 5), about two of the points $0, \varrho_1, \varrho_2, \infty$. We obtain thus six solutions which are entire functions. They differ from the six solutions (141), when the latter exist, only by periodic factors [cf. eq. (90)]. We have namely

$$(143) \quad \begin{cases} h_1''(x) = (1 - e^{2\pi i\beta_1})(1 - e^{2\pi i x})\, h_1(x), \\ h_2''(x) = (1 - e^{2\pi i\beta_2})(1 - e^{2\pi i x})\, h_2(x), \\ g_1''(x) = (1 - e^{2\pi i\beta_1})(1 - e^{-2\pi i(x+\beta_1+\beta_2)})\, g_1(x), \\ g_2''(x) = (1 - e^{2\pi i\beta_2})(1 - e^{-2\pi i(x+\beta_1+\beta_2)})\, g_2(x), \\ l''(x) = (1 - e^{2\pi i\beta_1})(1 - e^{2\pi i\beta_2})\, l(x), \\ m''(x) = (1 - e^{2\pi i x})(1 - e^{-2\pi i(x+\beta_1+\beta_2)})\, m(x). \end{cases}$$

Since the solutions on the left are valid for all values of $x$, $\beta_1$, and $\beta_2$, we will regard eqs. (143) as defining the solutions

* For a further discussion of these double loop circuits, see Barnes: *Messenger of Math.*, 34 (1904), pp. 52–71.

$h_1(x), h_2(x), g_1(x), g_2(x), l(x)$, and $m(x)$ whenever the integrals in (141) are not valid. We will assume for the present that neither $\beta_1$ nor $\beta_2$ is a negative integer, since some of the solutions are meaningless in that case; these excluded values will be discussed in § 10. We see that $l(x)$ is an entire function, while the other five solutions are analytic throughout the finite part of the plane except for possible poles at points where the periodic factors vanish.

Consider the solution $h_1(x)$ in (141); if we set $t = \varrho_1 \tau$, then $t - \varrho_1 = -\varrho_1(1-\tau)$ and

$$t - \varrho_2 = (\varrho_1 - \varrho_2)\left(1 - \frac{1-\tau}{1-\varkappa}\right) = -\varrho_2\left(1 - \frac{\tau}{\varkappa}\right),$$

where $\varkappa = \varrho_2/\varrho_1$.* Using the first form of $t - \varrho_2$, we have

$$h_1(x) = (-\varrho_1)^{\beta_1}(\varrho_1 - \varrho_2)^{\beta_2}\varrho_1^x \int_0^1 \tau^{x-1}(1-\tau)^{\beta_1}\left(1 - \frac{1-\tau}{1-\varkappa}\right)^{\beta_2} d\tau .$$

If $|1-\varkappa| > 1$, the last factor of the integrand can be expanded by the binomial theorem and the resulting series integrated term by term with the aid of eq. (89). If the restrictions $R(x) > 0$, $R(\beta_1) > -1$ are not satisfied, we may use the definition of $h_1(x)$ given in (143) and evaluate $h_1''(x)$ by means of eq. (90). In either case we find that

$$\begin{aligned} h_1(x) &= (-\varrho_1)^{\beta_1}(\varrho_1 - \varrho_2)^{\beta_2}\varrho_1^x \Big[B(x, \beta_1 + 1) \\ &\qquad - \frac{\beta_2}{1-\varkappa} B(x, \beta_1 + 2) + \cdots\Big] \\ &= (-\varrho_1)^{\beta_1}(\varrho_1 - \varrho_2)^{\beta_2}\varrho_1^x \frac{\Gamma(x)\Gamma(\beta_1+1)}{\Gamma(x+\beta_1+1)} \\ &\qquad \times\left(1 - \frac{\beta_2}{1-\varkappa}\,\frac{\beta_1+1}{x+\beta_1+1} + \cdots\right) \\ &= (-\varrho_1)^{\beta_1}(\varrho_1 - \varrho_2)^{\beta_2}\varrho_1^x B(x, \beta_1 + 1) \\ &\qquad \times F\left(\beta_1 + 1, -\beta_2, x + \beta_1 + 1, \frac{1}{1-\varkappa}\right), \end{aligned}$$

where $F$ denotes the hypergeometric function (100). Since the variable $x$ occurs only in the third argument of $F$, this

* We will suspend temporarily our convention that $|\varrho_1| \geq |\varrho_2|$.

hypergeometric series is included in the class of factorial series, defined by eq. (98). If $\varrho_1$ and $\varrho_2$ have values such that $|1-\varkappa| > 1$, it converges for all values of $x$ except $-\beta_1-1, -\beta_1-2, -\beta_1-3, \cdots$; even at these points $h_1(x)$ is analytic, since the series is multiplied by $B(x, \beta_1+1)$, which has a zero of the first order at these points. The only points where $h_1(x)$ is not analytic are $x = 0, -1, -2, \cdots$, where $B(x, \beta_1+1)$ has poles of the first order.

Using the second form of $t-\varrho_2$, we find in the same way that

$$h_1(x) = (-\varrho_1)^{\beta_1}(-\varrho_2)^{\beta_2}\varrho_1^{x} B(x, \beta_1+1) \times F\left(x, -\beta_2, x+\beta_1+1, \frac{1}{\varkappa}\right),$$

provided $|\varkappa| > 1$. Two other forms for $h_1(x)$ are obtained if we set $t = \varrho_1\varrho_2\tau/[\varrho_2-\varrho_1(1-\tau)]$, which may be written in either of the forms

$$-\frac{\varrho_1\varrho_2\tau}{\varrho_1-\varrho_2}\left(1-\frac{\tau}{1-\varkappa}\right)^{-1}, \qquad \varrho_1\tau\left(1-\frac{1-\tau}{\varkappa}\right)^{-1};$$

the corresponding forms of $t-\varrho_1$ and $t-\varrho_2$ are

$$-\varrho_1(1-\tau)\left(1-\frac{\tau}{1-\varkappa}\right)^{-1}, \qquad \frac{\varrho_1(\varrho_1-\varrho_2)}{\varrho_2}(1-\tau)\left(1-\frac{1-\tau}{\varkappa}\right)^{-1}$$

and

$$-\varrho_2\left(1-\frac{\tau}{1-\varkappa}\right)^{-1}, \qquad (\varrho_1-\varrho_2)\left(1-\frac{1-\tau}{\varkappa}\right)^{-1}$$

respectively. With these values we obtain the forms (b) and (d) of $h_1(x)$ in (144) below, which are valid if $|1-\varkappa| > 1$ and $|\varkappa| > 1$ respectively.

Similarly, if we use in $h_2(x)$ the transformations

$$t = \varrho_2\tau, \qquad t = \frac{\varrho_1\varrho_2\tau}{\varrho_1-\varrho_2(1-\tau)},$$

and evaluate the integral in the same ways as above, we obtain the four forms in (145). These results might be derived

from those for $h_1(x)$ by interchanging $\varrho_1$ and $\varrho_2$, $\beta_1$ and $\beta_2$, since this interchange leaves eq. (103) unaltered.

The other four solutions in (141) or (143) may be evaluated in a similar fashion. We obtain thus a set of 24 solutions of eq. (103) in terms of hypergeometric series, analogous to the 24 series which satisfy the hypergeometric differential equation. They are namely:

$$(144)\quad h_1(x)\begin{cases}\text{(a)}\ (-\varrho_1)^{\beta_1}(\varrho_1-\varrho_2)^{\beta_2}\varrho_1^x B(x,\beta_1+1)\\ \qquad\times F\left(\beta_1+1,-\beta_2,x+\beta_1+1,\dfrac{1}{1-\varkappa}\right),\\ \text{(b)}\ (-\varrho_1)^{\beta_1}(-\varrho_2)^{\beta_2}\left(\dfrac{\varrho_1\varrho_2}{\varrho_2-\varrho_1}\right)^x B(x,\beta_1+1)\\ \qquad\times F\left(x,x+\beta_1+\beta_2+1,x+\beta_1+1,\dfrac{1}{1-\varkappa}\right),\\ \text{(c)}\ (-\varrho_1)^{\beta_1}(-\varrho_2)^{\beta_2}\varrho_1^x B(x,\beta_1+1)\\ \qquad\times F\left(x,-\beta_2,x+\beta_1+1,\dfrac{1}{\varkappa}\right),\\ \text{(d)}\ -\varrho_2^{-\beta_1-1}(\varrho_1-\varrho_2)^{\beta_1+\beta_2+1}\varrho_1^{x+\beta_1}B(x,\beta_1+1)\\ \qquad\times F\left(x+\beta_1+\beta_2+1,\beta_1+1,x+\beta_1+1,\dfrac{1}{\varkappa}\right);\end{cases}$$

$$(145)\quad h_2(x)\begin{cases}\text{(a)}\ (-\varrho_2)^{\beta_2}(\varrho_2-\varrho_1)^{\beta_1}\varrho_2^x B(x,\beta_2+1)\\ \qquad\times F\left(\beta_2+1,-\beta_1,x+\beta_2+1,\dfrac{\varkappa}{\varkappa-1}\right),\\ \text{(b)}\ (-\varrho_1)^{\beta_1}(-\varrho_2)^{\beta_2}\left(\dfrac{\varrho_1\varrho_2}{\varrho_1-\varrho_2}\right)^x B(x,\beta_2+1)\\ \qquad\times F\left(x,x+\beta_1+\beta_2+1,x+\beta_2+1,\dfrac{\varkappa}{\varkappa-1}\right),\\ \text{(c)}\ (-\varrho_1)^{\beta_1}(-\varrho_2)^{\beta_2}\varrho_2^x B(x,\beta_2+1)\\ \qquad\times F(x,-\beta_1,x+\beta_2+1,\varkappa),\\ \text{(d)}\ -\varrho_1^{-\beta_2-1}(\varrho_2-\varrho_1)^{\beta_1+\beta_2+1}\varrho_2^{x+\beta_2}B(x,\beta_2+1)\\ \qquad\times F(x+\beta_1+\beta_2+1,\beta_2+1,x+\beta_2+1,\varkappa);\end{cases}$$

$$
(146)\quad g_1(x)\begin{cases}
\text{(a)}\ -(\varrho_1-\varrho_2)^{\beta_2}\,\varrho_1^{x+\beta_1}\,B(\beta_1+1,\,-x-\beta_1-\beta_2)\\
\qquad\times F\left(\beta_1+1,\,-\beta_2,\,1-x-\beta_2,\,\dfrac{\varkappa}{\varkappa-1}\right),\\
\text{(b)}\ -(\varrho_1-\varrho_2)^{x+\beta_1+\beta_2}\,B(\beta_1+1,\,-x-\beta_1-\beta_2)\\
\qquad\times F\left(1-x,\,-x-\beta_1-\beta_2,\,1-x-\beta_2,\,\dfrac{\varkappa}{\varkappa-1}\right),\\
\text{(c)}\ -(\varrho_1-\varrho_2)^{\beta_1+\beta_2+1}\,\varrho_1^{x-1}\,B(\beta_1+1,\,-x-\beta_1-\beta_2)\\
\qquad\times F(1-x,\,\beta_1+1,\,1-x-\beta_2,\,\varkappa),\\
\text{(d)}\ -\varrho_1^{x+\beta_1+\beta_2}\,B(\beta_1+1,\,-x-\beta_1-\beta_2)\\
\qquad\times F(-x-\beta_1-\beta_2,\,-\beta_2,\,1-x-\beta_2,\,\varkappa);
\end{cases}
$$

$$
(147)\quad g_2(x)\begin{cases}
\text{(a)}\ -(\varrho_2-\varrho_1)^{\beta_1}\,\varrho_2^{x+\beta_2}\,B(\beta_2+1,\,-x-\beta_1-\beta_2)\\
\qquad\times F\left(\beta_2+1,\,-\beta_1,\,1-x-\beta_1,\,\dfrac{1}{1-\varkappa}\right),\\
\text{(b)}\ -(\varrho_2-\varrho_1)^{x+\beta_1+\beta_2}\,B(\beta_2+1,\,-x-\beta_1-\beta_2)\\
\qquad\times F\left(1-x,\,-x-\beta_1-\beta_2,\,1-x-\beta_1,\,\dfrac{1}{1-\varkappa}\right),\\
\text{(c)}\ -(\varrho_2-\varrho_1)^{\beta_1+\beta_2+1}\,\varrho_2^{x-1}\,B(\beta_2+1,\,-x-\beta_1-\beta_2)\\
\qquad\times F\left(1-x,\,\beta_2+1,\,1-x-\beta_1,\,\dfrac{1}{\varkappa}\right),\\
\text{(d)}\ -\varrho_2^{x+\beta_1+\beta_2}\,B(\beta_2+1,\,-x-\beta_1-\beta_2)\\
\qquad\times F\left(-x-\beta_1-\beta_2,\,-\beta_1,\,1-x-\beta_1,\,\dfrac{1}{\varkappa}\right);
\end{cases}
$$

$$
(148)\quad l(x)\begin{cases}
\text{(a)}\ (-1)^{\beta_2+1}(\varrho_2-\varrho_1)^{\beta_1+\beta_2+1}\,\varrho_1^{x-1}\,B(\beta_1+1,\,\beta_2+1)\\
\qquad\times F(1-x,\,\beta_1+1,\,\beta_1+\beta_2+2,\,1-\varkappa),\\
\text{(b)}\ (-\varrho_1)^{-\beta_2-1}(\varrho_2-\varrho_1)^{\beta_1+\beta_2+1}\,\varrho_2^{x+\beta_2}\,B(\beta_1+1,\,\beta_2+1)\\
\qquad\times F(x+\beta_1+\beta_2+1,\,\beta_2+1,\,\beta_1+\beta_2+2,\,1-\varkappa),\\
\text{(c)}\ (-1)^{\beta_2+1}(\varrho_2-\varrho_1)^{\beta_1+\beta_2+1}\,\varrho_2^{x-1}\,B(\beta_1+1,\,\beta_2+1)\\
\qquad\times F\left(1-x,\,\beta_2+1,\,\beta_1+\beta_2+2,\,\dfrac{\varkappa-1}{\varkappa}\right),\\
\text{(d)}\ -(-\varrho_2)^{-\beta_1-1}(\varrho_1-\varrho_2)^{\beta_1+\beta_2+1}\,\varrho_1^{x+\beta_1}\,B(\beta_1+1,\beta_2+1)\\
\qquad\times F\left(x+\beta_1+\beta_2+1,\beta_1+1,\beta_1+\beta_2+2,\dfrac{\varkappa-1}{\varkappa}\right);
\end{cases}
$$

$$(149)\quad m(x)\begin{cases}\text{(a)}\ (-\varrho_1)^{\beta_1}(-\varrho_2)^{x+\beta_2}B(x,-x-\beta_1-\beta_2)\\ \qquad\times F(x,-\beta_1,-\beta_1-\beta_2,1-\varkappa),\\ \text{(b)}\ (-\varrho_1)^{x+\beta_1+\beta_2}B(x,-x-\beta_1-\beta_2)\\ \qquad\times F(-x-\beta_1-\beta_2,-\beta_2,-\beta_1-\beta_2,1-\varkappa),\\ \text{(c)}\ (-\varrho_2)^{\beta_2}(-\varrho_1)^{x+\beta_1}B(x,-x-\beta_1-\beta_2)\\ \qquad\times F\left(x,-\beta_2,-\beta_1-\beta_2,\dfrac{\varkappa-1}{\varkappa}\right),\\ \text{(d)}\ (-\varrho_2)^{x+\beta_1+\beta_2}B(x,-x-\beta_1-\beta_2)\\ \qquad\times F\left(-x-\beta_1-\beta_2,-\beta_1,-\beta_1-\beta_2,\dfrac{\varkappa-1}{\varkappa}\right).\text{*}\end{cases}$$

The convergence of the hypergeometric series depends on the fourth argument, which is a function of $\varkappa = \varrho_2/\varrho_1$. If $\varrho_1$ and $\varrho_2$ have values such that the fourth argument is less than 1 in absolute value the series converges for all values of $x$; if it is greater than 1 the series diverges The fourth argument cannot be equal to 1, since by hypothesis $\varrho_1$ and $\varrho_2$ are different from each other; it may however have its absolute value equal to 1; in this case the hypergeometric series (100) converges if $R(\gamma-\alpha-\beta) > -1$, and diverges if $R(\gamma-\alpha-\beta) \leq -1$. If the expression $\gamma-\alpha-\beta$ for one of the series above involves $x$, this means that the series converges in a certain half plane; if not, the convergence depends on the values of $\beta_1$ and $\beta_2$. The series break down when the third argument is zero or a negative integer, but, as pointed out in connection with $h_1(x)$ above, the solutions

* The transformations of the variable of integration needed in deriving the above forms for $h_1(x)$ and $h_2(x)$ have already been stated; for the other four solutions they are as follows.

$g_1(x)$: (a), (d), $t = \dfrac{\varrho_1}{\tau}$; (b), (c), $t = \dfrac{\varrho_1-\varrho_2\tau}{1-\tau}$;

$g_2(x)$: (a), (d), $t = \dfrac{\varrho_2}{\tau}$; (b), (c), $t = \dfrac{\varrho_2-\varrho_1\tau}{1-\tau}$;

$l(x)$: (a), (c), $t = \varrho_1-(\varrho_1-\varrho_2)\tau$; (b), (d), $t = \dfrac{\varrho_1\varrho_2}{\varrho_1-(\varrho_1-\varrho_2)\tau}$;

$m(x)$: (a), (d); $t = \dfrac{-\varrho_2\tau}{1-\tau}$; (b), (c), $t = \dfrac{-\varrho_1\tau}{1-\tau}$.

remain analytic at such points, and their values can be obtained from the series of beta functions from which the hypergeometric series were derived.

Under our convention that $|\varrho_1| \geqq |\varrho_2|$, i. e., $|\varkappa| \leqq 1$, at least one of the four series for $h_2(x)$ and $g_1(x)$ converges for all possible values of $\varrho_1$, $\varrho_2$, $\beta_1$, and $\beta_2$; for if $|\varrho_1| > |\varrho_2|$, the series (145c), (145d), (146c), (146d) all converge, while if $|\varrho_1| = |\varrho_2|$, the series (145d) and (146c) converge when $R(\beta_1+\beta_2) < 0$ and (145c) and (146d) when $R(\beta_1+\beta_2) > -2$.

In the case of the solutions $h_1(x)$ and $g_2(x)$, however, all of the series diverge if $\varrho_1$ and $\varrho_2$ have values such that $\varkappa$ lies inside both the circles $|\varkappa| = 1$ and $|1-\varkappa| = 1$. In this case series which are convergent, though of a less simple form, can be obtained as follows. In evaluating the integral form of $h_1(x)$, instead of setting $t = \varrho_1\tau$ as above, let us set $t = \varrho_1 \tau^{\frac{1}{\omega}}$; then

$$h_1(x) = (-\varrho_1)^{\beta_1}(\varrho_1-\varrho_2)^{\beta_2}\varrho_1^x \int_0^1 \tau^{\frac{x}{\omega}-1}\left(1-\tau^{\frac{1}{\omega}}\right)^{\beta_1}\left(1-\frac{1-\tau^{\frac{1}{\omega}}}{1-\varkappa}\right)^{\beta_2}\frac{d\tau}{\omega}.$$

If the third factor of the integrand is expanded by the binomial theorem, the resulting series converges inside a circle with its center at $\tau = 1$ and with the point $\tau = \varkappa^{\omega}$ on its circumference. If $\varkappa$ does not lie on the axis of reals between 0 and 1 (i. e., if $\arg\varrho_1 \neq \arg\varrho_2$), we can give $\omega$ a positive real value large enough so that $\varkappa^{\omega}$ lies outside the circle $|\tau-1| = 1$; if $\varkappa$ does lie on the positive axis of reals, it is necessary to give $\omega$ a complex value. The third factor will then be analytic inside a circle with center $\tau = 1$ and radius greater than 1. The second factor of the integrand is analytic everywhere in the $\tau$-plane except at $\tau = 1$. By the binomial theorem,

$$1-\tau^{\frac{1}{\omega}} = 1-[1-(1-\tau)]^{\frac{1}{\omega}} = \frac{1-\tau}{\omega}\left[1+\frac{\omega-1}{2!}\left(\frac{1-\tau}{\omega}\right)+\cdots\right],$$

which converges for $|1-\tau| < 1$. Substituting this in the second and third factors of the integrand, we have

$$h_1(x) = (-\varrho_1)^{\beta_1} (\varrho_1 - \varrho_2)^{\beta_2} \varrho_1^x$$
$$\times \int_0^1 \tau^{\frac{x}{\omega}-1} (1-\tau)^{\beta_1} [A_0 + A_1(1-\tau) + \cdots] \, d\tau,$$

where

$$A_0 = \frac{1}{\omega^{\beta_1+1}}, \quad A_1 = \frac{\beta_1(\omega - 1)}{2\,\omega^{\beta_1+2}} - \frac{\beta_2}{(1-\varkappa)\,\omega^{\beta_1+2}}, \text{ etc.}$$

The power series here converges in the neighborhood of $\tau = 1$;* but the function it represents has no singularities nearer to 1 than $\varkappa^\omega$; hence the radius of convergence must exceed 1. We can therefore integrate term by term if $R(x/\omega) > 0$, $R(\beta_1) > -1$, and obtain the series

$$h_1(x) = (-\varrho_1)^{\beta_1} (\varrho_1 - \varrho_2)^{\beta_2} \varrho_1^x \, \Gamma\left(\frac{x}{\omega}\right)$$
$$\cdot \left[A_0 \frac{\Gamma(\beta_1 + 1)}{\Gamma\left(\frac{x}{\omega} + \beta_1 + 1\right)} + A_1 \frac{\Gamma(\beta_1 + 2)}{\Gamma\left(\frac{x}{\omega} + \beta_1 + 2\right)} + \cdots\right].$$

The coefficient of $A_n$ is equal approximately for large values of $n$ to $(\beta_1 + n + 1)^{-\frac{x}{\omega}}$, as we see from (65); comparison with the absolutely convergent series

$$A_0 + A_1(1+\varepsilon) + A_2(1+\varepsilon)^2 + \cdots,$$

where $\varepsilon$ is a sufficiently small positive constant, shows that the series for $h_1(x)$ converges uniformly in the neighborhood of every value of $x$. It may be written as a factorial series:

$$(150)\quad \begin{cases} h_1(x) = (-\varrho_1)^{\beta_1} (\varrho_1 - \varrho_2)^{\beta_2} \varrho_1^x \, B\left(\frac{x}{\omega}, \beta_1 + 1\right) \\ \qquad \times \left[A_0 + A_1 \dfrac{\beta_1 + 1}{\frac{x}{\omega} + \beta_1 + 1}\right. \\ \qquad\qquad \left. + A_2 \dfrac{(\beta_1+1)(\beta_1+2)}{\left(\frac{x}{\omega} + \beta_1 + 1\right)\left(\frac{x}{\omega} + \beta_1 + 2\right)} + \cdots\right]. \end{cases}$$

* Cf. e. g. Knopp: *Theorie und Anwendung der unendlichen Reihen* (2nd ed.), **104**, p. 179.

Evaluating $g_2(x)$ in the same way, we have

$$
(151)\quad \begin{cases} g_2(x) = -(\varrho_2-\varrho_1)^{\beta_1}\,\varrho_2^{x+\beta_2}\,B\left(\dfrac{-x-\beta_1-\beta_2}{\omega},\ \beta_2+1\right) \\[2ex] \qquad\qquad \times\left[A_0'+A_1'\dfrac{\beta_2+1}{\dfrac{-x-\beta_1-\beta_2}{\omega}+\beta_2+1}+\cdots\right], \end{cases}
$$

where $A_i'$ differs from $A_i$ only by the interchange of $\beta_1$ and $\beta_2$.

We will now prove that the solutions $h_1(x)$, $h_2(x)$, $g_1(x)$, $g_2(x)$ are identical with the principal solutions $h_{11}(x)$, $h_{12}(x)$, $g_{11}(x)$, $g_{12}(x)$ obtained in § 6. From (144a) and (145a), in which the hypergeometric functions contain $x$ only in their third arguments, we see with the aid of (65) that if $x\to\infty$ in the right half plane

$$
\lim_{x\to\infty}[\varrho_1^{-x}x^{\beta_1+1}h_1(x)] = (-\varrho_1)^{\beta_1}(\varrho_1-\varrho_2)^{\beta_2}\Gamma(\beta_1+1),
$$
$$
\lim_{x\to\infty}[\varrho_2^{-x}x^{\beta_2+1}h_2(x)] = (-\varrho_2)^{\beta_2}(\varrho_2-\varrho_1)^{\beta_1}\Gamma(\beta_2+1).
$$

In case (144a) is divergent we may use (150), and in case (145a) is divergent we obtain the same limit from (145c) or (145d), in which the hypergeometric functions approach the binomial expansions of $(1-\varkappa)^{\beta_1}$ and $(1-\varkappa)^{-\beta_2-1}$ as $x\to\infty$. Hence if we write

$$
(152)\quad \begin{cases} s_1 = (-\varrho_1)^{\beta_1}(\varrho_1-\varrho_2)^{\beta_2}\Gamma(\beta_1+1), \\ s_2 = (-\varrho_2)^{\beta_2}(\varrho_2-\varrho_1)^{\beta_1}\Gamma(\beta_2+1), \end{cases}
$$

we see that $h_1(x)$ and $h_2(x)$ are represented asymptotically by the first term of $S_1(x)$ and $S_2(x)$ respectively in the right half plane. This is sufficient, as we saw at the end of § 6, to prove that $h_1(x) = h_{11}(x)$ and $h_2(x) = h_{12}(x)$.

Now let $x\to\infty$ in the left half plane; $(-x)\to\infty$ in the right half plane, and we have from (146a) and (147a)

$$
\lim_{x\to\infty}[\varrho_1^{-x}x^{\beta_1+1}g_1(x)] = (-\varrho_1)^{\beta_1}(\varrho_1-\varrho_2)^{\beta_2}\Gamma(\beta_1+1) = s_1,
$$
$$
\lim_{x\to\infty}[\varrho_2^{-x}x^{\beta_2+1}g_2(x)] = (-\varrho_2)^{\beta_2}(\varrho_2-\varrho_1)^{\beta_1}\Gamma(\beta_2+1) = s_2.
$$

If (146 a) and (147 a) are divergent we may use (146 c) or (146 d) and (151). Hence $g_1(x)$ and $g_2(x)$ are represented asymptotically by the first term of $S_1(x)$ and $S_2(x)$ in the left half plane, and this identifies them with $g_{11}(x)$ and $g_{12}(x)$ respectively.

The principal solutions may also be expressed by means of series of partial fractions.* Let $a$ be a point on the straight line joining 0 and $\varrho_1$, nearer to 0 than $\varrho_2$ is. The contour consisting of the straight line segment from 0 to $a$, a loop $k_1$ enclosing $\varrho_1$ but not $\varrho_2$, and the straight line segment from $a$ back to 0 is equivalent to the loop $l_1$ of Fig. 8, so if $R(x) > 0$ we have

$$\begin{aligned} h_1'(x) &- (1-e^{2\pi i\beta_1})\,h_1(x) \\ &= (1-e^{2\pi i\beta_1})\int_0^a t^{x-1}\,v(t)\,dt + \int_{k_1} t^{x-1}\,v(t)\,dt. \end{aligned}$$

The last integral is an entire function, which we will denote by $(1-e^{2\pi i\beta_1})E_1(x)$; then if $\beta_1$ is not an integer

$$\begin{aligned} h_1(x) = E_1(x) &+ (-\varrho_1)^{\beta_1}(-\varrho_2)^{\beta_2} \\ &\times \int_0^a t^{x-1}\left(1-\frac{t}{\varrho_1}\right)^{\beta_1}\left(1-\frac{t}{\varrho_2}\right)^{\beta_2} dt. \end{aligned}$$

If we expand the second and third factors of the integrand by the binomial theorem, the resulting series converge absolutely and uniformly along the path of integration, since $|a| < |\varrho_2|$. Multiplying them together and integrating term by term, we have

$$\begin{aligned} h_1(x) = E_1(x) + (-\varrho_1)^{\beta_1}(-\varrho_2)^{\beta_2}\,a^x\Bigg\{&\frac{1}{x} - \left(\frac{\beta_1}{\varrho_1}+\frac{\beta_2}{\varrho_2}\right)\frac{a}{x+1} \\ &+\left[\frac{\beta_1(\beta_1-1)}{2\varrho_1^2}+\frac{\beta_1\,\beta_2}{\varrho_1\,\varrho_2}+\frac{\beta_2(\beta_2-1)}{2\varrho_2^2}\right]\frac{a^2}{x+2}+\cdots\Bigg\}. \end{aligned}$$

It is evident from the way this series was derived that it converges if $R(x) > 0$. It also converges for all other values of $x$ except $0, -1, -2, \cdots$; for the convergence of a series

* Cf. Nörlund: *Differenzenrechnung*, p. 330.

depends on the terms after the $n$th, where $n$ is arbitrarily large; hence even if $R(x) < 0$ we can choose $n$ large enough so that $R(x+n) > 0$, and the terms after the $n$th are then obtained by integrating term by term a uniformly convergent series. The convergence is uniform with respect to $x$ in the neighborhood of any point except $0, -1, -2, \cdots$, so the series defines an analytic function of $x$; it is equal to $h_1(x)$ when $R(x) > 0$, but since both $h_1(x)$ and the series are analytic for all values of $x$ except $0, -1, -2, \cdots$, they are identically equal.

Similarly, if $b$ is a point on the straight line segment joining 0 and $\varrho_2$, and if $\beta_2$ is not an integer,

$$h_2(x) = E_2(x) + (-\varrho_1)^{\beta_1}(-\varrho_2)^{\beta_2} b^x \times \left[\frac{1}{x} - \left(\frac{\beta_1}{\varrho_1} + \frac{\beta_2}{\varrho_2}\right)\frac{b}{x+1} + \cdots\right],$$

where $E_2(x)$ is an entire function. We see from these partial fraction series that the residues of $h_1(x)$ and $h_2(x)$ at the poles $0, -1, -2, \cdots$, are the coefficients of the expansion of $v(t)$ about $t = 0$.

In the same way, if we take points $c$ and $d$ on the straight lines $\varrho_1 \infty$ and $\varrho_2 \infty$ at a distance from the origin greater than $|\varrho_1|$, we find that if $\beta_1$ and $\beta_2$ respectively are not integers

$$g_1(x) = E_1'(x) + c^{x+\beta_1+\beta_2}\left[\frac{1}{x+\beta_1+\beta_2} - \frac{\beta_1\varrho_1 + \beta_2\varrho_2}{c(x+\beta_1+\beta_2-1)} + \frac{\beta_1(\beta_1-1)\varrho_1^2 + 2\beta_1\beta_2\varrho_1\varrho_2 + \beta_2(\beta_2-1)\varrho_2^2}{2c^2(x+\beta_1+\beta_2-2)} + \cdots\right],$$

$$g_2(x) = E_2'(x) + d^{x+\beta_1+\beta_2}\left[\frac{1}{x+\beta_1+\beta_2} - \frac{\beta_1\varrho_1 + \beta_2\varrho_2}{d(x+\beta_1+\beta_2-1)} + \cdots\right],$$

where $E_1'(x)$ and $E_2'(x)$ are entire functions; these series converge uniformly in the neighborhood of every point except $-\beta_1-\beta_2, 1-\beta_1-\beta_2, \cdots$. We see that the residues of $g_1(x)$

and $g_2(x)$ at the poles $-\beta_1-\beta_2$, $1-\beta_1-\beta_2$, $\cdots$ are the coefficients of the expansion of $t^{-\beta_1-\beta_2}v(t)$ about $t=\infty$.

The series (148) and (149) for $l(x)$ and $m(x)$ are all divergent if $|1-\varkappa|>1$. In this case we can obtain convergent developments by expressing these solutions in terms of the principal solutions, as we shall see in the next section.

## § 8. Relations between the solutions. The fundamental periodic functions.

Since $h_1(x)$ and $h_2(x)$ are linearly independent, they form a fundamental system of solutions of eq. (103), and every solution can be expressed in the form $p_1(x)\,h_1(x)+p_2(x)\,h_2(x)$, where $p_1(x)$ and $p_2(x)$ are periodic functions. Likewise $g_1(x)$ and $g_2(x)$ form a fundamental system of solutions. The periodic functions by means of which $h_1(x)$ and $h_2(x)$ are expressed in terms of $g_1(x)$ and $g_2(x)$ and vice versa are of particular importance, and are called the *fundamental periodic functions*. We proceed to determine the explicit form of these and also of the periodic functions by means of which $l(x)$ and $m(x)$ are expressed in terms of the principal solutions.

Recalling our choice (131) of $\arg\varrho_1$ and $\arg\varrho_2$, let us suppose first that $\arg\varrho_1>\arg\varrho_2$, and consider the integral

$$\int t^{x-1}(t-\varrho_1)^{\beta_1}(t-\varrho_2)^{\beta_2}\,dt$$

over the closed contour $ABCDEFA$ of Fig. 9, made up of straight line segments and circular arcs. Since the contour contains no singular point of the integrand, the integral vanishes. If $R(x)>1$, $R(\beta_1)>0$, and $R(\beta_2)$

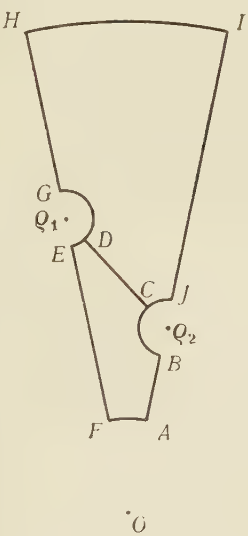


Fig. 9.


Fig. 10.

$>0$, we can let the radius of each of the circular arcs approach zero; the part of the integral contributed by the arcs then approaches zero, and we have in the limit (omitting the integrands)

$$\int_0^{\varrho_2}+\int_{\varrho_2}^{\varrho_1}-\int_0^{\varrho_1}=0.$$

Let us choose the arguments of the factors of the integrand so that $\arg t=\arg\varrho_2$ and $\arg(t-\varrho_2)=\arg\varrho_2+\pi$ on $AB$, and $\arg(t-\varrho_1)=\arg\varrho_1+\pi$ on $EF$; then by the definitions of the preceding section the three integrals are $h_2(x)$, $l(x)$, and $h_1(x)$ respectively, so we have

(153) $$l(x)=h_1(x)-h_2(x).$$

Similarly, the integral over the closed contour $CDGHIJC$ of Fig. 9 vanishes. If $R(x+\beta_1+\beta_2)<0$, $R(\beta_1)>0$, and $R(\beta_2)>0$, we can let the radius of the arcs $DG$ and $JC$ approach 0, and that of $HI$ increase indefinitely; then we have in the limit

$$\int_{\varrho_2}^{\varrho_1}-\int_\infty^{\varrho_1}+\int_\infty^{\varrho_2}=0.$$

If we choose the arguments of $t$, $t-\varrho_1$, and $t-\varrho_2$ so that they are the same on $CD$ as in the previous case, the first integral is $l(x)$ and the third $g_2(x)$; but in the second integral the argument of $t-\varrho_1$ is $\arg\varrho_1+2\pi$, so it is not equal to $g_1(x)$, in which $\arg(t-\varrho_1)=\arg\varrho_1$, but what $g_1(x)$ becomes when $\arg(t-\varrho_1)$ is increased by $2\pi$, namely $e^{2\pi i\beta_1}g_1(x)$: hence

(154) $$l(x)=e^{2\pi i\beta_1}g_1(x)-g_2(x).$$

If $\arg\varrho_1<\arg\varrho_2$, we need merely to interchange $\varrho_1$ and $\varrho_2$ in Fig. 9 and integrate over the same contours; eq. (153) remains unaltered and (154) becomes

(155) $$l(x)=g_1(x)-e^{2\pi i\beta_2}g_2(x).$$

Since both sides of eqs. (153), (154), and (155) are analytic for all values of $x$, $\beta_1$, and $\beta_2$ (aside from the poles of the

solutions), these results must be identities, so we can drop the restrictions placed on $x$, $\beta_1$, and $\beta_2$ above.

If in Fig. 9 we increase the argument of $\varrho_2$ until it is equal to that of $\varrho_1$, we get Fig. 10, from which we see that

$$\int_0^{\varrho_2} + \int_{\varrho_2}^{\varrho_1} = \int_0^{\varrho_1}, \qquad \int_{\varrho_2}^{\varrho_1} - \int_{\infty}^{\varrho_1} = -\int_{\infty}^{\varrho_2};$$

if we start with $\arg t = \arg \varrho_2$ $(= \arg \varrho_1)$ and $\arg(t-\varrho_1) = \arg(t-\varrho_2) = \arg \varrho_2 + \pi$, these equations show that eqs. (153) and (154) continue to hold when $\arg \varrho_1 = \arg \varrho_2$.*

Returning to the case $\arg \varrho_1 > \arg \varrho_2$, let us integrate over the contour $ABCDEFA$ of Fig. 11. If $R(x) > 1$, $R(\beta_2) > 0$, and $R(x+\beta_1+\beta_2) < 0$, we can let the radius of the arcs $AF$ and $BC$ approach zero, and that of $DE$ increase indefinitely; then in the limit

$$\int_0^{\varrho_2} - \int_{\infty}^{\varrho_2} - \int^{\infty} = 0.$$

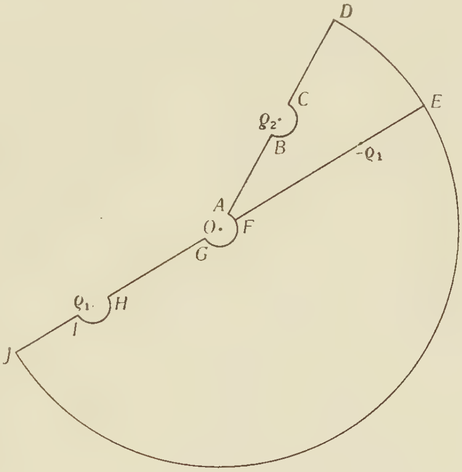


Fig. 11.

If we take $\arg t = \arg \varrho_2$ and $\arg(t-\varrho_2) = \arg \varrho_2 + \pi$ on $AB$, with $\arg(t-\varrho_1)$ between $\arg \varrho_1 + \pi$ and $\arg \varrho_1 + 2\pi$, the three integrals are respectively $h_2(x)$, $e^{2\pi i\beta_2}\, g_2(x)$, and $m(x)$; hence

(156) $$m(x) = h_2(x) - e^{2\pi i\beta_2}\, g_2(x).$$

Integrate now over the contour $GHIJEFG$, taking $\arg t = \arg \varrho_1$ and $\arg(t-\varrho_1) = \arg \varrho_1 + \pi$ on $GH$, with

* According to our definition of the integer $\lambda$ in § 6, the case $\arg \varrho_1 = \arg \varrho_2$ is the limiting case of $\arg \varrho_1 > \arg \varrho_2$, as we have taken it here, and not of $\arg \varrho_1 < \arg \varrho_2$.

$\arg(t-\varrho_2)$ between $\arg\varrho_2$ and $\arg\varrho_2+\pi$. If $R(x)>1$, $R(\beta_1)>0$, and $R(x+\beta_1+\beta_2)<0$, we can let the radius of the arcs $FG$ and $HI$ approach zero, and that of $JE$ increase indefinitely, so

$$\int_0^{\varrho_1}-\int_\infty^{\varrho_1}-\int_0^\infty = 0.$$

The three integrals are respectively $h_1(x)$, $g_1(x)$, and $e^{2\pi ix}\,m(x)$; hence

$$e^{2\pi ix}\,m(x) = h_1(x)-g_1(x). \tag{157}$$

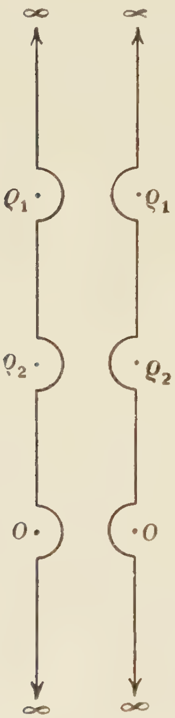

Fig. 12

If $\arg\varrho_1<\arg\varrho_2$, we find similarly that

$$m(x) = h_1(x)-e^{2\pi i\beta_1}\,g_1(x), \tag{158}$$

$$e^{2\pi ix}\,m(x) = h_2(x)-g_2(x). \tag{159}$$

Equations (156)–(159) are all identities in $x$, $\beta_1$, and $\beta_2$, so we can drop the restrictions on these quantities.

If we increase $\arg\varrho_2$ in Fig. 11 until it is equal to $\arg\varrho_1$, the two contours used above become like those in Fig. 12, and we see that (156) and (157) continue to hold.

If we equate the values of $l(x)$ in eqs. (153) and (154) and the values of $m(x)$ in eqs. (156) and (157), we obtain the equations

$$\begin{aligned} h_1(x)-h_2(x) &= e^{2\pi i\beta_1}\,g_1(x)-g_2(x),\\ h_1(x)-g_1(x) &= e^{2\pi ix}\,h_2(x)\\ &\quad -e^{2\pi i(x+\beta_2)}\,g_2(x). \end{aligned}$$

Eliminating in turn $h_2(x)$ and $h_1(x)$ between these, we obtain the equations

$$\tag{160} \left\{ \begin{aligned} h_1(x) &= \frac{1-e^{2\pi i(x+\beta_1)}}{1-e^{2\pi ix}}\,g_1(x)+\frac{(1-e^{2\pi i\beta_2})\,e^{2\pi ix}}{1-e^{2\pi ix}}\,g_2(x),\\ h_2(x) &= \frac{1-e^{2\pi i\beta_1}}{1-e^{2\pi ix}}\,g_1(x)+\frac{1-e^{2\pi i(x+\beta_2)}}{1-e^{2\pi ix}}\,g_2(x); \end{aligned} \right.$$

which express the solutions of the first principal system in terms of those of the second in the case $\arg\varrho_1 \geq \arg\varrho_2$, or $\lambda = 1$. Proceeding similarly with eqs. (153), (155), (158), and (159), we find that

$$
(161) \quad \begin{cases} h_1(x) = \dfrac{1-e^{2\pi i(x+\beta_1)}}{1-e^{2\pi ix}}\, g_1(x) + \dfrac{1-e^{2\pi i\beta_2}}{1-e^{2\pi ix}}\, g_2(x), \\[2ex] h_2(x) = \dfrac{(1-e^{2\pi i\beta_1})\, e^{2\pi ix}}{1-e^{2\pi ix}}\, g_1(x) + \dfrac{1-e^{2\pi i(x+\beta_2)}}{1-e^{2\pi ix}}\, g_2(x) \end{cases}
$$

in the case $\arg\varrho_1 < \arg\varrho_2$, or $\lambda = 0$. These two pairs of equations may be combined in the form

$$
(162) \quad \begin{cases} h_1(x) = \dfrac{1-e^{2\pi i(x+\beta_1)}}{1-e^{2\pi ix}}\, g_1(x) + \dfrac{c_2\, e^{2\lambda\pi ix}}{1-e^{2\pi ix}}\, g_2(x), \\[2ex] h_2(x) = \dfrac{c_1\, e^{2(1-\lambda)\pi ix}}{1-e^{2\pi ix}}\, g_1(x) + \dfrac{1-e^{2\pi i(x+\beta_2)}}{1-e^{2\pi ix}}\, g_2(x), \end{cases}
$$

where

$$
(163) \qquad c_1 = 1-e^{2\pi i\beta_1}, \qquad c_2 = 1-e^{2\pi i\beta_2}.
$$

If we increase the argument of $\varrho_2$ by $2k\pi$, where $k$ is any integer, $g_2(x)$ and $h_2(x)$ have to be replaced by $e^{-2k\pi ix}\, g_2(x)$ and $e^{-2k\pi ix}\, h_2(x)$; but $\lambda$ is diminished by $k$, so eqs. (162) remain true for all values of $\lambda$, i. e., for arbitrary determinations of $\arg\varrho_1$ and $\arg\varrho_2$.

The periodic functions in (162) are what we called at the beginning of this section the *fundamental periodic functions.* They form a matrix

$$
(164) \quad P(x) = \begin{Vmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{Vmatrix} = \begin{Vmatrix} \dfrac{1-e^{2\pi i(x+\beta_1)}}{1-e^{2\pi ix}} & \dfrac{c_1\, e^{2(1-\lambda)\pi ix}}{1-e^{2\pi ix}} \\[2ex] \dfrac{c_2\, e^{2\lambda\pi ix}}{1-e^{2\pi ix}} & \dfrac{1-e^{2\pi i(x+\beta_2)}}{1-e^{2\pi ix}} \end{Vmatrix}.
$$

The matrix solutions $H(x)$ and $G(x)$ of eq. (108) are connected by the relation $H(x) = G(x)\, P(x)$, which is equivalent

to eqs. (162), as we see by expanding the matrices. If we multiply this on the right by $P^{-1}(x)$, we have $G(x) = H(x)P^{-1}(x)$, so the matrix of periodic functions by means of which $G(x)$ is expressed in terms of $H(x)$ is the inverse of $P(x)$; its elements are

$$\bar{p}_{11}(x) = \frac{1 - e^{2\pi i(x+\beta_2)}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}}, \quad \bar{p}_{12}(x) = \frac{-c_1 e^{2(1-\lambda)\pi i x}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}} .$$

$$\bar{p}_{21}(x) = \frac{-c_2 e^{2\lambda\pi i x}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}}, \quad \bar{p}_{22}(x) = \frac{1 - e^{2\pi i(x+\beta_1)}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}} .$$

Hence

$$(165)\begin{cases} g_1(x) = \dfrac{1 - e^{2\pi i(x+\beta_2)}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}} h_1(x) + \dfrac{(e^{2\pi i\beta_2} - 1)\, e^{2\lambda\pi i x}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}} h_2(x), \\[2ex] g_2(x) = \dfrac{(e^{2\pi i\beta_1} - 1)\, e^{2(1-\lambda)\pi i x}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}} h_1(x) + \dfrac{1 - e^{2\pi i(x+\beta_1)}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}} h_2(x). \end{cases}$$

These equations can also be obtained by solving eqs. (162) for $g_1(x)$ and $g_2(x)$.

Eq. (153) expresses $l(x)$ in terms of $h_1(x)$ and $h_2(x)$ for both $\lambda = 1$ and $\lambda = 0$. Eqs. (154) and (155) can be combined into the equation

$$(166) \qquad l(x) = e^{2\lambda\pi i\beta_1} g_1(x) - e^{2(1-\lambda)\pi i\beta_2} g_2(x),$$

which holds for both values of $\lambda$.* We observe that the periodic functions by which $l(x)$ is expressed in terms of the principal solutions are merely constants.

If in eq. (156) we replace $h_2(x)$ by its value from (160), and in eq. (158) replace $h_1(x)$ by its value from (161), we obtain in both cases the equation

$$(167) \qquad m(x) = \frac{c_1}{1 - e^{2\pi i x}} g_1(x) + \frac{c_2}{1 - e^{2\pi i x}} g_2(x).$$

* It does not, however, hold for general values of $\lambda$, unlike eqs. (162) and (165); we have in fact defined the solution $l(x)$ only for $\lambda = 1$ or $0$. Similar remarks apply to eq. (168).

Similarly, eliminating $g_1(x)$ and $g_2(x)$ from (156) and (158) with the aid of (165), we obtain two equations which can be combined in the form

$$(168)\quad m(x) = \frac{c_1\, e^{2\lambda\pi i\beta_2}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}}\, h_1(x) + \frac{c_2\, e^{2(1-\lambda)\pi i\beta_1}}{1 - e^{2\pi i(x+\beta_1+\beta_2)}}\, h_2(x),$$

which holds for both values of $\lambda$.

The formulas of this section enable us to express $l(x)$ and $m(x)$ in terms of convergent series when $\varrho_1$ and $\varrho_2$ have values such that the series in (148) and (149) all diverge. We have already observed that $l(x)$ is an entire function; this is also evident from eq. (153), since $h_1(x)$ and $h_2(x)$ have the same poles with the same residues. We see from eqs. (156)–(159) that $m(x)$ has two sets of poles, one extending to the left and one to the right, namely $x = 0, -1, -2, \cdots$, and $x = -\beta_1 - \beta_2,\ 1 - \beta_1 - \beta_2, \cdots$.

We will now obtain analytic expressions for the intermediate solutions, whose existence and asymptotic properties were proved in § 5. As we saw there, we can take $y'_{12}(x) = h_{12}(x)$, $y'_{22}(x) = h_{22}(x)$, i. e., $y'_{12}(x) = h_2(x)$, $y'_{22}(x) = h_2(x+1)$. In case $|\varrho_1| = |\varrho_2|$, we can also take $y'_{11}(x) = h_1(x)$, $y'_{21}(x) = h_1(x+1)$, so we need to consider here only the case $|\varrho_1| > |\varrho_2|$.

Let us write

$$y'_{11}(x) = p(x)\, h_1(x) + q(x)\, h_2(x),$$

where $p(x)$ and $q(x)$ are periodic functions to be determined, or

$$1 = p(x)\,\frac{h_1(x)}{y'_{11}(x)} + q(x)\,\frac{h_2(x)}{y'_{11}(x)}.$$

Let $x \to \infty$ along a ray in the first quadrant parallel to the axis of reals and sufficiently far above it to avoid the singularities of $y'_{11}(x)$. Along this ray $y'_{11}(x) \sim S_1(x)$, by § 5, so

$$1 \sim p(x) + q(x)\left(\frac{\varrho_2}{\varrho_1}\right)^x x^{\beta_1-\beta_2}\left(\frac{s_2}{s_1} + \cdots\right);$$

but the factor $(\varrho_2/\varrho_1)^x$ approaches zero exponentially along this ray; hence $p(x) \sim 1$, i. e., $p(x) = 1$, so

$$y_{11}'(x) = h_1(x) + q(x)\, h_2(x). \tag{169}$$

Now express $h_1(x)$ and $h_2(x)$ in terms of $g_1(x)$ and $g_2(x)$ by (162):

$$\begin{aligned} y_{11}'(x) = {} & [p_{11}(x) + q(x)\, p_{12}(x)]\, g_1(x) \\ & + [p_{21}(x) + q(x)\, p_{22}(x)]\, g_2(x). \end{aligned} \tag{170}$$

Let $x \to \infty$ along a ray in the second quadrant nearly parallel to the negative axis of reals; $y_{11}'(x) \sim S_1(x)$, so dividing by $y_{11}'(x)$ we have

$$\begin{aligned} 1 \sim {} & [p_{11}(x) + q(x)\, p_{12}(x)] \\ & + [p_{21}(x) + q(x)\, p_{22}(x)] \left(\frac{\varrho_2}{\varrho_1}\right)^x x^{\beta_1 - \beta_2} \left(\frac{s_2}{s_1} + \cdots\right). \end{aligned}$$

But $(\varrho_2/\varrho_1)^x$ becomes infinite exponentially if the angle which the ray makes with the axis of reals is small enough; hence we must have

$$p_{21}(x) + q(x)\, p_{22}(x) = 0,$$

or, by (164),

$$q(x) = -\frac{p_{21}(x)}{p_{22}(x)} = -\frac{c_2\, e^{2\lambda\pi i x}}{1 - e^{2\pi i (x+\beta_2)}}.$$

Using this in (169) and (170), we have

$$\begin{aligned} y_{11}'(x) &= h_1(x) - \frac{c_2\, e^{2\lambda\pi i x}}{1 - e^{2\pi i (x+\beta_2)}}\, h_2(x) \\ &= \frac{1 - e^{2\pi i (x+\beta_1+\beta_2)}}{1 - e^{2\pi i (x+\beta_2)}}\, g_1(x). \end{aligned} \tag{171}$$

These equalities hold sufficiently far above (or below) the axis of reals; but since the second and third members are analytic near the axis of reals, they may be regarded as defining $y_{11}'(x)$ there. In the same way we find for the other pair of intermediate solutions that $y_{11}'(x) = g_1(x)$ and

$$\bar{y}'_{12}(x) = g_2(x) + \frac{c_1 e^{2(1-\lambda)\pi i x}}{1-e^{2\pi i(x+\beta_2)}} g_1(x) = \frac{1-e^{2\pi i x}}{1-e^{2\pi i(x+\beta_2)}} h_2(x).$$

These formulas enable us to study the properties of the intermediate solutions near the axis of reals. We see that $y'_{11}(x)$ and $\bar{y}'_{12}(x)$ are both analytic except for simple poles at all points congruent to $x = -\beta_2$; $y'_{11}(x)$ has zeros at points congruent to $x = -\beta_1 - \beta_2$ on the left, and $y'_{12}(x)$ has zeros at $x = 1, 2, 3, \cdots$.

We can express $y'_{11}(x)$ in terms of a hypergeometric series by using its second form in (171). By eq. (59),

$$\frac{e^{\pi i(\beta_1+1)}\Gamma(x+\beta_2)}{\Gamma(x+\beta_1+\beta_2+1)} = \frac{1-e^{2\pi i(x+\beta_1+\beta_2)}}{1-e^{2\pi i(x+\beta_2)}} \frac{\Gamma(-x-\beta_1-\beta_2)}{\Gamma(1-x-\beta_2)};$$

hence, by (146 a),

$$y'_{11}(x) = (-\varrho_1)^{\beta_1}(\varrho_1-\varrho_2)^{\beta_2}\varrho_1^x B(x+\beta_2, \beta_1+1)$$
$$\times F\left(\beta_1+1, -\beta_2, 1-x-\beta_2, \frac{\varkappa}{\varkappa-1}\right).$$

The other three forms of $g_1(x)$ in (146) may also be used. Similarly,

$$\bar{y}'_{12}(x) = -(\varrho_2-\varrho_1)^{\beta_1}\varrho_2^{x+\beta_2} B(-x-\beta_2, \beta_2+1)$$
$$\cdot F\left(\beta_2+1, -\beta_1, x+\beta_2+1, \frac{\varkappa}{\varkappa-1}\right).$$

## § 9. Asymptotic forms.

With the aid of the fundamental periodic functions, we are now in a position to study the asymptotic forms of the principal solutions in the entire plane.

We have seen (§ 4) that $h_2(x)$ [$= h_{12}(x)$] is represented asymptotically by $S_2(x)$ for large values of $x$ in the sector $-\pi < \arg x < \pi$; let us investigate now how this solution behaves as $x \to \infty$ along a ray parallel to the negative axis of reals. By (162) and (164),

$$h_2(x) = p_{12}(x)\, g_1(x) + p_{22}(x)\, g_2(x),$$

whence

$$(172)\qquad h_2(x) \sim p_{12}(x)\, S_1(x) + p_{22}(x)\, S_2(x)$$

along such a ray. If $\varrho_1 < \varrho_2$, the second term is the dominant one, since $(\varrho_2/\varrho_1)^x$ increases exponentially along the ray, so in this case $h_2(x) \sim p_{22}(x)\, S_2(x)$. Suppose first that the ray is above the axis of reals, and write $x = u + iv$ $(u < 0,\ v > 0)$; we see from (164) that $p_{22}(x)$ differs from 1 by a quantity of the order of $e^{-2\pi v}$; hence we can write

$$h_2(x) \sim [1 + \varpi(x)]\, S_2(x),$$

where $\varpi(x)$ is a periodic function such that $|\varpi(x)| < Me^{-2\pi v}$ for large values of $v$, $M$ being a constant. Since $e^{-2\pi v}$ approaches zero very rapidly as $v$ increases, we have $h_2(x) \sim S_2(x)$ with as close a degree of approximation as we wish by taking the ray far enough above the axis of reals.

If the ray is below the axis of reals, $p_{22}(x)$ differs from $e^{2\pi i\beta_2}$ by a quantity of the order of $e^{2\pi v}$; hence, just as above, we have $h_2(x) \sim e^{2\pi i\beta_2}\, S_2(x)$ to an arbitrary degree of approximation if we take the ray sufficiently far below the axis of reals. Here $S_2(x)$ denotes the determination of the series for $g_2(x)$, i. e., for the argument of $x$ near $\pi$; if we take the argument near $-\pi$, we have $h_2(x) \sim S_2(x)$.

Hence when $\varrho_1 < \varrho_2$ we may say that $h_2(x) \sim S_2(x)$ along any ray in the whole plane, except one which is near the negative axis of reals throughout its length. A similar argument shows that $g_1(x) \sim S_1(x)$ along any ray except one which is near to the positive axis of reals or to the row of poles at $x = -\beta_1 - \beta_2,\ 1 - \beta_1 - \beta_2, \cdots$.

If $\varrho_1 = \varrho_2$, the two terms in (172) are of the same order of magnitude, and $h_2(x)$ is represented by their sum along any ray parallel to the negative axis of reals. Similarly,

$$g_1(x) \sim p_{11}(x)\, S_1(x) + p_{21}(x)\, S_2(x)$$

along rays parallel to the positive axis of reals in this case.

Consider now the behavior in the left half plane of the solution $h_1(x)$, which we know is represented asymptotically in the right half plane by $S_1(x)$. By (162) and (164),

$$h_1(x) = p_{11}(x)\, g_1(x) + p_{21}(x)\, g_2(x);$$

hence along any ray in the second quadrant

$$h_1(x) \sim S_1(x) + c_2\, e^{2\lambda\pi i x}\, S_2(x). \tag{173}$$

The dominant term depends on the factors $\varrho_1^x$, $e^{2\lambda\pi i x}\varrho_2^x$, or, if we divide by the first one and use only the exponents, on

$$0,\quad 2\pi i\left(\lambda + \frac{\log\varrho_2 - \log\varrho_1}{2\pi i}\right)x.$$

Writing $x = u + iv$ $(u < 0,\ v > 0)$ and using (132), we see that the real part of the second exponent is $-bu - 2\pi a v$; this is negative if $v/|u| > b/2\pi a$ and positive if $v/|u| < b/2\pi a$; in the former case the first term dominates, in the latter case the second. Hence there is a *critical ray* in the second quadrant, making the angle

$$\varphi = \tan^{-1}\frac{b}{2\pi a} = \tan^{-1}\frac{\log|\varrho_1| - \log|\varrho_2|}{2\lambda\pi + \arg\varrho_2 - \arg\varrho_1} \tag{174}$$

with the negative axis of reals, at which the asymptotic form changes; we have namely $h_1(x) \sim S_1(x)$ if $\arg x < \pi - \varphi$, and $h_1(x) \sim c_2\, e^{2\lambda\pi i x}\, S_2(x)$ if $\pi - \varphi < \arg x < \pi$. Along the critical ray itself the two terms on the right in (173) are of equal order of magnitude, so $h_1(x)$ is represented by their sum.

In the third quadrant,

$$h_1(x) \sim e^{2\pi i\beta_1}\, S_1(x) - c_2\, e^{2(\lambda-1)\pi i x}\, S_2(x);$$

here $S_1(x)$ and $S_2(x)$ denote the determinations of the series for which the argument of $x$ is between $\pi$ and $3\pi/2$; if we diminish the argument by $2\pi$, we have

$$h_1(x) \sim S_1(x) - c_2\, e^{-2\pi i\beta_2}\, e^{2(\lambda-1)\pi i x}\, S_2(x).$$

The dominant term depends on the exponents

$$0, \quad 2\pi i\left(\lambda-1+\frac{\log\varrho_2-\log\varrho_1}{2\pi i}\right)x;$$

the real part of the second is $-bu+2\pi(1-a)v$ $(u<0, v<0)$; this is negative if $v/u>b/2\pi(1-a)$, and positive if $v/u<b/2\pi(1-a)$, so we have another critical ray in the third quadrant, making the angle

$$(175)\quad \psi=\tan^{-1}\frac{b}{2\pi(1-a)}=\tan^{-1}\frac{\log|\varrho_1|-\log|\varrho_2|}{2(1-\lambda)\pi+\arg\varrho_1-\arg\varrho_2}$$

with the negative axis of reals. The first term dominates if $\arg x>-\pi+\psi$, and the second if $-\pi<\arg x<-\pi+\psi$.* Summing up these results, we see that

$$(176)\quad h_1(x)\sim\begin{cases}-c_2e^{-2\pi i\beta_2}e^{2(\lambda-1)\pi ix}S_2(x), & -\pi<\arg x<-\pi+\psi;\\ S_1(x), & -\pi+\psi<\arg x<\pi-\varphi;\\ c_2e^{2\lambda\pi ix}S_2(x), & \pi-\varphi<\arg x<\pi.\end{cases}$$

Similarly, by expressing $g_2(x)$ in terms of $h_1(x)$ and $h_2(x)$ by (165), we find that

$$g_2(x)\sim\begin{cases}-c_1e^{2(\lambda-1)\pi ix}S_1(x), & 0<\arg x<\psi;\\ S_2(x), & \psi<\arg x<2\pi-\varphi;\\ c_1e^{-2\pi i\beta_2}e^{-2\lambda\pi ix}S_1(x), & 2\pi-\varphi<\arg x<2\pi.\end{cases}$$

Here $\varphi$ and $\psi$ denote the same angles as above.

The asymptotic forms of $l(x)$ and $m(x)$ in the right half plane can be obtained from their expressions in terms of $h_1(x)$ and $h_2(x)$ [eqs. (153), (168)], and in the left half plane from their expressions in terms of $g_1(x)$ and $g_2(x)$ [eqs. (154), (155), (167)]. We find that when $\lambda=1$,

* If $\arg\varrho_1=\arg\varrho_2$, we have $a=1$, so $\psi=\pi/2$ and the second term dominates everywhere in the third quadrant except along rays parallel to the negative axis of imaginaries, where the two terms are of equal order of magnitude. This agrees with our discussion of this case in § 6.

$$
l(x) \sim \begin{cases} S_1(x), & -\pi + \psi_1 < \arg x < \psi_1; \\ -S_2(x), & \psi_1 < \arg x < \pi + \psi_1; \end{cases}
$$

$$
m(x) \sim \begin{cases} c_1 e^{2\pi i \beta_2} S_1(x), & 0 < \arg x < \psi_1; \\ c_2 S_2(x), & \psi_1 < \arg x < \pi; \\ \quad c_2 e^{-2\pi i x} S_2(x), & \pi < \arg x < \pi + \psi_1; \\ \cdot c_1 e^{-2\pi i x} S_1(x), & \pi + \psi_1 < \arg x < 2\pi; \end{cases}
$$

here $\psi_1$ denotes the value of $\psi$ [eq. (175)] for $\lambda = 1$. When $\lambda = 0$,

$$
l(x) \sim \begin{cases} S_1(x), & -\varphi_0 < \arg x < \pi - \varphi_0; \\ -S_2(x), & -\pi - \varphi_0 < \arg x < -\varphi_0; \end{cases}
$$

$$
m(x) \sim \begin{cases} c_1 S_1(x), & 0 < \arg x < \pi - \varphi_0; \\ c_2 S_2(x), & \pi - \varphi_0 < \arg x < \pi; \\ -c_2 e^{-2\pi i x} S_2(x), & \pi < \arg x < 2\pi - \varphi_0; \\ -c_1 e^{-2\pi i (x+\beta_2)} S_1(x), & 2\pi - \varphi_0 < \arg x < 2\pi; \end{cases}
$$

$\varphi_0$ denotes the value of $\varphi$ [eq. (174)] for $\lambda = 0$. These solutions have critical rays in the first and third quadrants if $\lambda = 1$, and in the second and fourth if $\lambda = 0$. We note that the results above give the asymptotic form of $l(x)$ in the complete neighborhood of $x = \infty$; in the case of $m(x)$ none of the forms hold near the positive or negative axis of reals.

## § 10. Reducible equations.

A linear homogeneous difference equation with rational coefficients is said to be *reducible* if it has a solution in common with an equation of the same type of lower order.* In accordance with this definition, a hypergeometric difference equation is reducible if it has a solution $y(x)$ which satisfies the first order equation

$$
(38) \qquad y(x+1) - r(x)\, y(x) = 0 \quad [r(x) \text{ rational}],
$$

which we studied in Chap. II.

* Wallenberg und Guldberg: *Theorie der linearen Differenzengleichungen*, p. 114.

If eq. (103) has a solution in common with this, it is satisfied by both the expressions (74), since any solution of eq. (38) differs from each of these only by a periodic factor. These two solutions are represented asymptotically by the series (44) in the sectors $-\pi < \arg x < \pi$ and $0 < \arg x < 2\pi$ respectively. But this series must be either $S_1(x)$ or $S_2(x)$ (apart from a constant factor), on account of the uniqueness of the latter, and hence the solutions (74) are proportional either to $h_1(x)$ and $g_1(x)$ or to $h_2(x)$ and $g_2(x)$. Since the solutions (74) differ from each other only by a periodic factor, it follows from (162) that either $c_1$ or $c_2$ is zero.

This necessary condition is also sufficient. For let $c_1 = 0$; then by (162) $h_2(x) = p_{22}(x)\, g_2(x)$. Consider the ratio

$$\varphi(x) = \frac{h_2(x+1)}{h_2(x)} = \frac{g_2(x+1)}{g_2(x)};$$

from the asymptotic forms of $h_2(x)$ and $g_2(x)$ we see that

$$\varphi(x) \sim \frac{S_2(x+1)}{S_2(x)} = \varrho_2\left(1 - \frac{\beta_2 + 1}{x} + \cdots\right)$$

in the complete neighborhood of $x = \infty$. It follows, as we saw in § 6, Chap. I, that this series is convergent for large values of $x$. The function $\varphi(x)$ is therefore analytic outside of a certain circle; inside this circle it has no singularities except a finite number of poles; it must therefore be a rational function $r(x)$. This shows that $h_2(x)$ and $g_2(x)$ satisfy an equation of the form (38). In the same way we see that if $c_2 = 0$, $h_1(x)$ and $g_1(x)$ satisfy an equation of this form.

THEOREM. *A necessary and sufficient condition that eq.* (103) *be reducible is that either $c_1$ or $c_2$ be zero.*

We see from (163) that $c_1$ or $c_2$ is zero when and only when $\beta_1$ or $\beta_2$ is an integer. In § 7, however, we excluded negative integral values of $\beta_1$ and $\beta_2$, so we need to investigate these values.

If we regard $h_1(x)$ and $g_1(x)$ as functions of $\beta_1$, they are analytic in general, but have simple poles at $\beta_1 = -1, -2, -3, \cdots$. If we multiply them by $1 - e^{2\pi i \beta_1}$ we get the solutions $h_1'(x)$ and $g_1'(x)$ [eq. (142)], which are analytic for these values of $\beta_1$; these solutions are represented asymptotically by $S_1(x)$ if we multiply $s_1$ [eq. (152)] by the same factor. This multiplication by $1 - e^{2\pi i \beta_1}$ is equivalent to replacing $\Gamma(\beta_1 + 1)$ by $\overline{\Gamma}(\beta_1 + 1)$, as we see from eq. (57). We will define $h_1'(x)$ and $g_1'(x)$ as the principal solutions in place of $h_1(x)$ and $g_1(x)$ when $\beta_1$ is a negative integer, and similarly define

$$h_2'(x) = (1 - e^{2\pi i \beta_2})\, h_2(x), \qquad g_2'(x) = (1 - e^{2\pi i \beta_2})\, g_2(x)$$

as the principal solutions in place of $h_2(x)$ and $g_2(x)$ when $\beta_2$ is a negative integer.

If we multiply the first equation in (162) by $1 - e^{2\pi i \beta_1}$ $(= c_1)$, we get the identity

$$h_1'(x) = \frac{1 - e^{2\pi i (x + \beta_1)}}{1 - e^{2\pi i x}}\, g_1'(x) + \frac{c_1\, c_2\, e^{2\lambda \pi i x}}{1 - e^{2\pi i x}}\, g_2(x);$$

if now we let $\beta_1$ approach any negative integer, $c_1 \to 0$ and the equation becomes $h_1'(x) = g_1'(x)$. From this equality we infer as above that the ratio $h_1'(x+1)/h_1'(x) = g_1'(x+1)/g_1'(x)$ is a rational function, so eq. (103) is reducible. A similar argument shows that if $\beta_2$ is a negative integer, then $c_2 = 0$ and $h_2'(x) = g_2'(x)$, so again eq. (103) is reducible. This completes the proof of the theorem above, and also of the following one.

THEOREM. *A necessary and sufficient condition that eq.* (103) *be reducible is that either* $\beta_1$ *or* $\beta_2$ *be an integer.*

The forms of the principal solutions of a reducible equation are readily obtained from eqs. (144)–(147). Thus if $\beta_1$ is zero or a positive integer, all of the terms of the hypergeometric series in (145a) and (147a) are zero except the first $\beta_1 + 1$, so $h_2(x)$ and $g_2(x)$ have the respective forms

$$\varrho_2^x \frac{\Gamma(x)}{\Gamma(x + \beta_2 + 1)} R(x), \qquad \varrho_2^x \frac{\Gamma(-x - \beta_1 - \beta_2)}{\Gamma(1 - x - \beta_1)} R'(x),$$

where $R(x)$ and $R'(x)$ are rational functions. The denominators of $R(x)$ and $R'(x)$ may be combined with the gamma functions in the denominators, giving us

$$\varrho_2^x \frac{\Gamma(x)}{\Gamma(x+\beta_1+\beta_2+1)} P(x), \qquad \varrho_2^x \frac{\Gamma(-x-\beta_1-\beta_2)}{\Gamma(1-x)} P'(x),$$

where $P(x)$ and $P'(x)$ are polynomials of degree $\beta_1$. These polynomials can differ from each other only by a constant factor (namely $e^{\pi i(\beta_1+\beta_2+1)}$), since the ratio of $g_2(x)$ to $h_2(x)$ is a periodic function; the form of the periodic function, obtained from (59), is $(1-e^{2\pi i x})/(1-e^{2\pi i(x+\beta_2)})$, which agrees with (165) for $\beta_1$ an integer.

If $\beta_1$ is a negative integer, we see from (144a) that

$$h_1'(x) = \varrho_1^x \frac{\Gamma(x)}{\Gamma(x+\beta_1+1)} R(x),$$

where $R(x)$ is a rational function, and if the denominator of $R(x)$ is combined with $\Gamma(x+\beta_1+1)$ we get $\Gamma(x)$, which cancels out; $h_1'(x)$ is therefore equal to $\varrho_1^x P(x)$, where $P(x)$ is a polynomial of degree $-\beta_1-1$. As we saw above, $g_1'(x) = h_1'(x)$ in this case. Similar results are obtained when $\beta_2$ is an integer.

The discussion above shows that if $\beta_1$ is a positive integer (including zero) or $\beta_2$ a negative integer, the solutions $h_2(x)$ and $g_2(x)$ satisfy an equation of the form (38), and that if $\beta_1$ is a negative integer or $\beta_2$ a positive integer the solutions $h_1(x)$ and $g_1(x)$ satisfy such an equation. The other pair of principal solutions in each case do not in general satisfy any first order equation of this type; this is true even if $\beta_1$ is a positive integer *and* $\beta_2$ a negative integer or vice versa. If, however, $\beta_1$ and $\beta_2$ are both positive integers or both negative integers, all the principal solutions satisfy equations of the form (38); in this case the equation is called *completely reducible*. The gamma functions cancel out, and each solution is equal to $\varrho_i^x$ multiplied by a rational function. Thus for the equation

$$(x+3)\,y(x+2)-(x+3)\,y(x+1)-2\,x\,y(x) = 0,$$

in which $\varrho_1 = 2$, $\varrho_2 = -1$, $\beta_1 = 0$, $\beta_2 = 1$, the principal solutions are

$$h_1(x) = g_1(x) = \frac{2^x(3x+1)}{x(x+1)}, \quad h_2(x) = g_2(x) = \frac{(-1)^x}{x(x+1)}.$$

## § 11. The Riemann problem.

In the theory of linear differential equations Riemann proposed the problem of determining a differential equation of given type which should have prescribed monodromic group constants. An analogous problem for systems of linear difference equations of the first order with polynomial coefficients was proposed by Birkhoff* and later solved by him.†

Following Birkhoff, we will give the name *characteristic constants* to the roots $\varrho_1$, $\varrho_2$ of the characteristic equation (101), the exponential constants $\beta_1$, $\beta_2$ in the formal series (106), and the constants $c_1$, $c_2$ of the fundamental periodic functions (164). The question may now be asked whether it is possible to find a hypergeometric difference equation of the form (99) which has prescribed values for these six constants. That this is not in general the case is evident from the fact that eq. (99) contains only five independent constants, namely the ratios of the six numbers $a_2, b_2, a_1, b_1, a_0, b_0$; these ratios will in general be determined if five conditions are imposed. Let us ask therefore whether any five of the characteristic constants can be prescribed arbitrarily.

Consider the normal form (103); this involves only four constants: $\varrho_1$, $\varrho_2$, $\beta_1$, $\beta_2$. It is obvious, then, that a hypergeometric difference equation can be found with arbitrary values for the first four characteristic constants,‡ namely eq. (103) itself; the two remaining constants $c_1$ and $c_2$ are expressible in terms of these four, as we see from (163).

* *Trans. Amer. Math. Soc.*, 12 (1911), p. 284.

† *Proc. Amer. Academy of Arts and Sciences*, 49 (1913), pp. 553–559. Cf. also Nörlund: *Comptes Rendus*, 156 (1913), pp. 200–203.

‡ The choice of $\varrho_1$ and $\varrho_2$ must of course be limited to finite values different from each other and from zero.

This is not the most general situation, however. In deriving eq. (103) from eq. (102) we eliminated the constant $\beta_3$, and a reference to the transformation by which this was done shows that if we replace $x$ by $x+\beta_3$ in any solution of (103), the resulting function is a solution of (102). The fundamental periodic functions for (102) are likewise obtained from (164) by replacing $x$ by $x+\beta_3$; the numerator of $p_{12}(x+\beta_3)$ is $c_1\, e^{2(1-\lambda)\pi i(x+\beta_3)}$, which may be written $c_1' e^{2(1-\lambda)\pi i x}$, where $c_1' = c_1 e^{2(1-\lambda)\pi i\beta_3}$; this is the constant for eq. (102) corresponding to $c_1$ for eq. (103). Similarly the other constant for eq. (102) is $c_2' = c_2\, e^{2\lambda\pi i\beta_3}$. By giving the proper value to $\beta_3$, $c_1'$ can be made to take on any prescribed value if $\lambda = 0$, and $c_2'$ any prescribed value if $\lambda = 1$. If we remove the restrictions (131) made in § 6 on the arguments of $\varrho_1$ and $\varrho_2$, so that $\lambda$ may have any integral value, we see that *either* $c_1'$ or $c_2'$ may be given a prescribed value; the other one will then be uniquely determined. The first four characteristic constants are the same for eq. (102) as for eq. (103), since the characteristic equation is the same for both, and since the exponential factors $x^{\beta_1-1}$ and $x^{-\beta_2-1}$ in the series (106) are not changed when $x$ is replaced by $x+\beta_3$. We conclude, then, that *of the six characteristic constants $\varrho_1, \varrho_2, \beta_1, \beta_2, c_1, c_2$, the first four and either one of the last two may be assigned arbitrary values; the remaining one will then be uniquely determined. The hypergeometric equation which possesses these characteristic constants has the form* (102), *where*

$$\beta_3 = \frac{\log c_1 - \log(1 - e^{2\pi i\beta_1})}{2(1-\lambda)\pi i} \quad \text{or} \quad \frac{\log c_2 - \log(1 - e^{2\pi i\beta_2})}{2\lambda\pi i}.$$

## Chapter IV.

# The hypergeometric equation: irregular cases.

### § 1. The case $\varrho_2 = 0$.

In Chap. III we studied the solutions of the hypergeometric difference equation (99) under the hypothesis that the roots $\varrho_1$ and $\varrho_2$ of the characteristic equation

$$a_2 \varrho^2 + a_1 \varrho + a_0 = 0 \tag{101}$$

are finite, distinct from each other, and different from zero. In Chap. IV we will investigate how the theory must be modified when these conditions are not satisfied. In the present section we will study the case where one root is equal to zero, while the other is finite and different from zero; this means $a_0 = 0$, $a_1 \neq 0$, $a_2 \neq 0$.* We will denote by $\varrho$ (without subscript) the non-zero root, i. e., $\varrho = -a_1/a_2$. We may assume that $b_0 \neq 0$, since otherwise eq. (99) is equivalent to one of the first order.

To reduce eq. (99) to a normal form suitable for the present case, write

$$\frac{b_2}{a_2} = \beta + \gamma + 2,$$

$$\frac{b_1}{a_2} = -\varrho(\gamma + 1) - \sigma,$$

$$\frac{b_0}{a_2} = \varrho\sigma;$$

these equations determine the constants $\beta$, $\gamma$, and $\sigma$ uniquely. Eq. (99) becomes

$$(x+\beta+\gamma+2)\,y(x+2) - [\varrho(x+\gamma+1)+\sigma]\,y(x+1) + \varrho\sigma y(x) = 0;$$

* The case where one root is zero and one infinite is considered in § 2.

if we set $x+\gamma = x'$ and $y(x'-\gamma) = f(x')$, then $f(x)$ satisfies the equation

$$(177) \quad (x+\beta+2)\, y(x+2) - [\varrho(x+1)+\sigma]\, y(x+1) + \varrho\sigma y(x) = 0,$$

which we will take as our normal form.

This equation is satisfied formally by two power series in $1/x$, which may be obtained by setting

$$y(x) = x^{ax} b^x x^d \left(s + \frac{s'}{x} + \frac{s''}{x^2} + \cdots\right),$$

and determining the constants $a, b, d, s'/s, s''/s, \cdots$ as in § 2, Chap. III (cf. also § 1, Chap. II), namely

$$(178) \quad \begin{cases} S_1(x) = \varrho^x x^{-\beta-1} \left(s_1 + \dfrac{s_1'}{x} + \dfrac{s_1''}{x^2} + \cdots\right) \\ S_2(x) = x^{-x} \sigma^x e^x x^{-\frac{1}{2}} \left(s_2 + \dfrac{s_2'}{x} + \dfrac{s_2''}{x_2} + \cdots\right), \end{cases}$$

where

$$\frac{s_1'}{s_1} = (\beta+1)\left(\frac{\sigma}{\varrho} - \frac{\beta}{2}\right),$$

$$\frac{s_1''}{s_1} = \frac{(\beta+1)(\beta+2)}{2}\left[\frac{\sigma^2}{\varrho^2} + \frac{\sigma}{\varrho}(1-\beta) + \frac{\beta(3\beta+1)}{12}\right], \text{ etc.};$$

$$\frac{s_2'}{s_2} = -\frac{\beta\sigma}{\varrho} - \frac{1}{12},$$

$$\frac{s_2''}{s_2} = \frac{\beta(\beta-1)\sigma^2}{2\varrho^2} + \frac{13\beta\sigma}{12\varrho} + \frac{1}{288}, \text{ etc.};$$

$s_1$ and $s_2$ are of course arbitrary.

To obtain a system of two equations of the first order equivalent to the single equation (177), set $y(x) = y_1(x)$, $y(x+1) = y_2(x)$; then $y_1(x)$ and $y_2(x)$ satisfy the system

$$(179) \quad \begin{cases} y_1(x+1) = y_2(x), \\ y_2(x+1) = -\dfrac{\varrho\sigma}{x+\beta+2}\, y_1(x) + \dfrac{\varrho(x+1)+\sigma}{x+\beta+2}\, y_2(x). \end{cases}$$

which may be written as a matrix equation

(180) $$Y(x+1) = R(x)\, Y(x),$$
where

$$R(x) = \begin{Vmatrix} 0 & 1 \\ -\dfrac{\varrho\,\sigma}{x+\beta+2} & \dfrac{\varrho\,(x+1)+\sigma}{x+\beta+2} \end{Vmatrix}.$$

This matrix equation is satisfied formally by the matrix of series

$$S(x) = \begin{Vmatrix} \varrho^x x^{-\beta-1}\left(s_{11} + \dfrac{s'_{11}}{x} + \cdots\right) & x^{-x}\sigma^x e^x x^{-\frac{1}{2}}\left(s_{12} + \dfrac{s'_{12}}{x} + \cdots\right) \\ \varrho^x x^{-\beta-1}\left(s_{21} + \dfrac{s'_{21}}{x} + \cdots\right) & x^{-x}\sigma^x e^x x^{-\frac{1}{2}}\left(0 + \dfrac{s'_{22}}{x} + \cdots\right) \end{Vmatrix},$$

where $s_{11} = s_1$, $s'_{11} = s'_1$, $s_{12} = s_2$, $s'_{12} = s'_2$, $s_{21} = \varrho s_1$, $s'_{21} = \varrho\,(s'_1 - \beta s_1 - s_1)$, $s'_{22} = \sigma s_2$, $s''_{22} = \sigma s'_2$, etc. The determinant of $S(x)$ has the form

$$|S(x)| = x^{-x}(\varrho\,\sigma\, e)^x x^{-\beta-\frac{3}{2}}\left(d + \frac{d'}{x} + \cdots\right),$$

where $d = -\varrho s_1 s_2$, etc. The inverse of $S(x)$ is

$$S^{-1}(x) = \begin{Vmatrix} \varrho^{-x} x^{\beta+1}\left(0 + \dfrac{\sigma'_{11}}{x} + \cdots\right) & \varrho^{-x} x^{\beta+1}\left(\sigma_{12} + \dfrac{\sigma'_{12}}{x} + \cdots\right) \\ x^x \sigma^{-x} e^{-x} x^{\frac{1}{2}}\left(\sigma_{21} + \dfrac{\sigma'_{21}}{x} + \cdots\right) & x^x \sigma^{-x} e^{-x} x^{\frac{1}{2}}\left(\sigma_{22} + \dfrac{\sigma'_{22}}{x} + \cdots\right) \end{Vmatrix}.$$

where $\sigma'_{11} = -\sigma/\varrho s_1$, $\sigma_{12} = 1/\varrho s_1$, $\sigma_{21} = 1/s_2$, $\sigma_{22} = -1/\varrho s_2$, etc.

Eq. (180) has precisely the same form as (108), and is satisfied formally by the two infinite products (113). If we let $T(x)$ denote the matrix obtained from $S(x)$ by using only the first $k$ terms of each series and form the products $H_n(x)$ and $G_n(x)$ as in (114), we can apply the argument of § 4, Chap. III to prove the following existence theorem.

THEOREM. *Both elements of the second column of* $H_n(x)$ *converge uniformly as* $n \to \infty$ *to limit functions* $h_{12}(x)$, $h_{22}(x)$ *which are analytic in the entire finite part of the plane, and form a solution of the system* (179). *The determinant of* $H_n(x)$ *also converges to a limit function* $D(x)$ *analytic in the finite part of the plane. For large values of* $x$ *these limit functions are represented asymptotically by* $s_{12}(x)$, $s_{22}(x)$, *and* $|S(x)|$ *respectively in the sector* $-\pi < \arg x < \pi$.

*Similarly, both elements of the first column of* $G_n(x)$ *converge uniformly to limit functions* $g_{11}(x)$, $g_{21}(x)$ *which are analytic except for poles and form a second solution of* (179). *The determinant of* $G_n(x)$ *also converges to a limit function* $\bar{D}(x)$ *analytic except for poles. For large values of* $x$ *these limit functions are represented asymptotically by* $s_{11}(x)$, $s_{21}(x)$, *and* $|S(x)|$ *respectively in the sector* $0 < \arg x < 2\pi$.

The details of the proof are practically identical with those of the general case, except that $\varrho_1$ and $\varrho_2$ in the formulas are replaced by $\varrho$ and $e\sigma/x$, and $\beta_1 - \beta_2$ by $\beta - \frac{1}{2}$. In connection with (117), the fact that $\varrho_2$ is a function of $x+r$, $x+s$, etc. causes no difficulty, since we may write

$$\varrho_2^{x+t} = \left(\frac{e\sigma}{x+t}\right)^{x+t} = \left(\frac{e\sigma}{x}\right)^{x+t} e^{-t}\left(1 + \frac{a}{x} + \cdots\right),$$

and combine the last two factors with the corresponding $\varphi$. The limits $h_{12}(x)$, $h_{22}(x)$, and $D(x)$ are entire functions, while $g_{11}(x)$, $g_{21}(x)$, and $\bar{D}(x)$ in general have poles at the points $-1-\beta$, $-\beta$, $1-\beta$, $\cdots$;* $D(x)$ and $\bar{D}(x)$ have the respective forms

$$-\varrho s_1 s_2 \sqrt{2\pi}\, \frac{\varrho^x \sigma^x}{\Gamma(x+\beta+\frac{1}{2})}, \qquad -\varrho s_1 s_2 \sqrt{2\pi}\, \frac{\varrho^x \sigma^x}{\Gamma(x+\beta+\frac{1}{2})}.$$

The gap caused by the divergence of one column of $H_n(x)$ and $G_n(x)$ can be filled exactly as in the general case, and by evaluating the sum as an infinite series we obtain two

* $g_{11}(x)$ does not have a pole at $-1-\beta$; cf. footnote, p. 80.

pairs of intermediate solutions with the same properties as before. The details of the proof in § 5, Chap. III require only slight modification apart from the replacement of $\varrho_1$, $\varrho_2$, and $\beta_1 - \beta_2$ by $\varrho$, $e\sigma/x$, and $\beta - \frac{1}{2}$ in the formulas.

Solutions corresponding to the principal solutions of the general case are obtained as in § 6, Chap. III. We take for two of them the solutions $h_{12}(x)$, $h_{22}(x)$ and $g_{11}(x)$, $g_{21}(x)$ obtained above, and define two other solutions $h_{11}(x)$, $h_{21}(x)$ and $g_{12}(x)$, $g_{22}(x)$ by eqs. (128) and (136), where $\lambda$ and $\lambda'$ are integers; the angle $\varepsilon$ which the lines $A\infty$ and $B\infty$ of the contour $L$ make with the axis of reals may have any value $< \pi/2$ in this case. The function $h_{11}(x)$ is analytic in the entire finite part of the plane. Its asymptotic form in the first and fourth quadrants is given by (129) and (134), and in both the second term is the dominant one, on account of the factor $x^{-x}$ contained in the first term, except perhaps along a ray parallel to the axis of imaginaries. Hence $h_{11}(x) \sim s_{11}(x)$ and also $h_{21}(x) \sim s_{21}(x)$ in the sector $-\pi/2 < \arg x < \pi/2$; this is true regardless of the value of the integer $\lambda$, so there are an infinite number of solutions with this property. Any one of them can be combined with the solution $h_{12}(x)$, $h_{22}(x)$ to form a matrix solution $H(x)$ of eq. (180); the determinant of this matrix is equal to $D(x)$. Similarly, $g_{12}(x)$ and $g_{22}(x)$ are analytic except for poles at $-1-\beta$, $-\beta$, $1-\beta$, $\cdots$, and $g_{12}(x) \sim s_{12}(x)$, $g_{22}(x) \sim s_{22}(x)$ in the sector $\pi/2 < \arg x < 3\pi/2$, regardless of the value of $\lambda'$. This solution may be combined with $g_{11}(x)$, $g_{21}(x)$ to form a matrix solution $G(x)$, whose determinant is $\bar{D}(x)$.

We can prove by the same argument as in the general case that the solutions $h_{12}(x)$, $h_{22}(x)$ and $g_{11}(x)$, $g_{21}(x)$ are unique. The other two solutions, however, are clearly not unique, since the integers $\lambda$ and $\lambda'$ are entirely arbitrary; the difficulty is similar to that which we met with in the general case when the arguments of $\varrho_1$ and $\varrho_2$ are equal. Let us examine the asymptotic form of these solutions in the direction of the axis of imaginaries, with a view to finding the values of $\lambda$ and $\lambda'$ which give us solutions whose

properties correspond as closely as possible to those of the principal solutions in the general case.

As we see from (129), the asymptotic form of $h_{11}(x)$ in the upper half plane is determined by the factors

$$x^{-x}\,\sigma^x\,e^x\,e^{2\lambda\pi i x}, \qquad \varrho^x,$$

or, if we divide by the second one and use only the exponents, by

$$(-\log x + \log\sigma + 1 + 2\lambda\pi i - \log\varrho)\,x, \quad 0.$$

The real part of the first exponent is

$$(-\log|x| + \log|\sigma| + 1 - \log|\varrho|)\,u \\ \qquad -2\pi\left(\lambda + \frac{\arg\sigma - \arg\varrho - \arg x}{2\pi}\right)v.$$

Keep $u$ fixed, and let $v \to +\infty$; since $v$ increases more rapidly than $\log|x|$, this ultimately becomes negative if

$$\lambda > \frac{\arg\varrho - \arg\sigma + \arg x}{2\pi},$$

even if $u$ is negative, or, since $\arg x \to \pi/2$ as $v \to +\infty$, if

$$\lambda > \frac{\arg\varrho - \arg\sigma}{2\pi} + \frac{1}{4}.$$

If $\lambda$ is less than this value the real part of the first exponent ultimately becomes positive.

In the lower half plane, as we see from (134), $\lambda$ is replaced by $\lambda - 1$, so the real part of the first exponent is

$$(-\log|x| + \log|\sigma| + 1 - \log|\varrho|)\,u \\ \qquad -2\pi\left(\lambda - 1 + \frac{\arg\sigma - \arg\varrho - \arg x}{2\pi}\right)v.$$

Keep $u$ fixed, and let $v \to -\infty$; this ultimately becomes negative if

$$\lambda < 1 + \frac{\arg \varrho - \arg \sigma + \arg x}{2\pi},$$

even if $u$ is negative, or, since $\arg x \to -\pi/2$ as $v \to -\infty$, if

$$\lambda < \frac{\arg \varrho - \arg \sigma}{2\pi} + \frac{3}{4}.$$

If $\lambda$ is greater than this value, the real part of the first exponent ultimately becomes positive.

Let us take for $\lambda$ *the smallest integer which exceeds* $(\arg \varrho - \arg \sigma)/2\pi + \frac{1}{4}$, and write

$$\lambda - \left(\frac{\arg \varrho - \arg \sigma}{2\pi} + \frac{1}{4}\right) = a; \tag{181}$$

then $0 < a \leqq 1$. With this value of $\lambda$, $h_{11}(x) \sim s_{11}(x)$ along every ray parallel to the positive axis of imaginaries, and in case $a < \frac{1}{2}$ also along every ray parallel to the negative axis of imaginaries. (If $a = \frac{1}{2}$, $h_{11}(x) \sim s_{11}(x)$ along rays parallel to the negative axis of imaginaries and to the right of it.)

A similar discussion of $g_{12}(x)$ shows that it is represented asymptotically by $s_{12}(x)$ along rays parallel to the positive axis of imaginaries if

$$\lambda' > \frac{\arg \sigma - \arg \varrho}{2\pi} - \frac{1}{4}.$$

and along rays parallel to the negative axis of imaginaries if

$$\lambda' < \frac{\arg \sigma - \arg \varrho}{2\pi} - \frac{1}{4}.$$

Let us define $\lambda'$ as *the largest integer which is less than* $(\arg \sigma - \arg \varrho)/2\pi + \frac{1}{4}$, and write

$$\lambda' + a' = \frac{\arg \sigma - \arg \varrho}{2\pi} + \frac{1}{4} \quad (0 < a' \leqq 1).$$

Then $g_{12}(x) \sim s_{12}(x)$ along every ray parallel to the negative axis of imaginaries, and in case $a' < \frac{1}{2}$ also along every ray

parallel to the positive axis of imaginaries. (If $a' = \frac{1}{2}$, $g_{12}(x) \sim s_{12}(x)$ along rays parallel to the positive axis of imaginaries and to the left of it.) The solutions (128) and (136) in which these values of $\lambda$ and $\lambda'$ are used we will define as the principal solutions of (179).

By choosing suitable determinations of $\arg \varrho$ and $\arg \sigma$ we may always have

$$\arg \varrho - \pi < \arg \sigma - \frac{\pi}{2} \leqq \arg \varrho + \pi;$$

then

$$-\frac{1}{2} \leqq \frac{\arg \varrho - \left(\arg \sigma - \frac{\pi}{2}\right)}{2\pi} < \frac{1}{2},$$

and

$$0 < \frac{\arg \sigma - \arg \varrho}{2\pi} + \frac{1}{4} \leqq 1.$$

so $\lambda = 1$ if $\arg \varrho \geqq \arg \sigma - \pi/2$ and $\lambda = 0$ if $\arg \varrho < \arg \sigma - \pi/2$, while $\lambda' = 0$ in both cases. We will limit ourselves to this choice.

We will now prove that if $a \leqq \frac{1}{2}$, the solution $h_{11}(x)$, $h_{21}(x)$ in which the above value of $\lambda$ is used is the only solution which is represented asymptotically by $s_{11}(x)$, $s_{21}(x)$ in the closed sector $-\pi/2 < \arg(x - \alpha) \leqq \pi/2$, where $\alpha$ is any constant.

Assume that a different solution $h'_{11}(x)$, $h'_{21}(x)$ has this property. We can write

$$h'_{11}(x) = p_1(x)\, h_{11}(x) + p_2(x)\, h_{12}(x);$$

dividing by $h'_{11}(x)$, and letting $x \to \infty$ along any line parallel to the positive axis of reals, we see that

$$1 \sim p_1(x) + p_2(x)\, x^{-x} \sigma^x e^x \varrho^{-x} x^{\beta + \frac{1}{2}} \left(\frac{s_2}{s_1} + \cdots\right).$$

This shows that $p_1(x) \sim 1$, i. e., $p_1(x) = 1$, since $x^{-x}$ approaches zero so rapidly. Hence

$$p_2(x) = \frac{h'_{11}(x) - h_{11}(x)}{h_{12}(x)};$$

$p_2(x)$ is single-valued and analytic in a period strip sufficiently far to the right, except possibly at the ends of the strip, where we see from the asymptotic forms that

$$\lim_{x \to \infty} p_2(x)\, x^{-x} \sigma^x e^x \varrho^{-x} x^{\beta + \frac{1}{2}} = 0.$$

Write $x^{-x} = \left(e^{-\frac{\pi i}{2}} x\right)^{-x} e^{-\frac{\pi i x}{2}}$; the modulus of $\left(e^{-\frac{\pi i}{2}} x\right)^{-x}$ is $e^{-u \log |x| + v\left(\arg x - \frac{\pi}{2}\right)}$; the second term in the exponent approaches the value $-u$ at the upper end of the strip, since

$$\lim_{\arg x \to \frac{\pi}{2}} v\left(\arg x - \frac{\pi}{2}\right) = \lim_{\theta \to \frac{\pi}{2}} u \tan\theta \left(\theta - \frac{\pi}{2}\right)$$

$$= \lim_{\theta \to \frac{\pi}{2}} u \frac{\theta - \frac{\pi}{2}}{\cot\theta} = -u.$$

Hence at the upper end of the strip the modulus of $\left(e^{-\frac{\pi i}{2}} x\right)^{-x}$ behaves like $e^{-u(\log|x|+1)} = e^{-u} |x|^{-u}$, and $v$ is approximately constant, since we are dealing with a strip of unit width. Accordingly

$$\lim_{x \to \infty} p_2(x)\, e^{-\frac{\pi i x}{2}} \sigma^x e^x \varrho^{-x} x^{\beta + \frac{1}{2}} |x|^u = 0$$

at the upper end of the strip. Set $p_2(x) = q(z)$, where $z = e^{2\pi i x}$; then

$$\lim_{z \to 0} q(z)\, z^{\frac{1}{4} + \frac{1 + \log\sigma - \log\varrho}{2\pi i}} \varphi(\log z) = 0$$

or

$$\lim_{z \to 0} q(z)\, z^{-\frac{1}{4} + \frac{\arg\sigma - \arg\varrho}{2\pi}} z^{\frac{b}{2\pi i}} \varphi(\log z) = 0,$$

where

$$b \qquad 1 + \log|\sigma - \log \varrho$$

and $\varphi(\log z)$ denotes a function which behaves like a power of $\log z$.* By (181),

$$\lim_{z \to 0} q(z) z^{-\lambda+a} z^{\frac{b}{2\pi i}} \varphi(\log z) = 0.$$

Since $0 < a \leq \frac{1}{2}$, we see that $q(z)z^{-\lambda}$ cannot have a pole of as high as the first order at $z = 0$.

At the lower end of the strip, write $x^{-x} = \left(e^{\frac{\pi i}{2}} x\right)^{-x} e^{\frac{\pi i x}{2}}$: the modulus of the first factor behaves like $e^{-u}|x|^{-u}$, so

$$\lim_{x \to \infty} p_2(x) e^{\frac{\pi i x}{2}} \sigma^x e^x \varrho^{-x} x^{\beta+\frac{1}{2}} |x|^{-u} = 0,$$

or, in terms of $z$,

$$\lim_{z \to \infty} q(z) z^{\frac{1}{4}+\frac{\arg\sigma-\arg\varrho}{2\pi}} z^{\frac{b}{2\pi i}} \varphi(\log z) = 0,$$

$$\lim_{z \to \infty} q(z) z^{-\lambda+u+\frac{1}{2}} z^{\frac{b}{2\pi i}} \varphi(\log z) = 0.$$

This shows that $q(z)z^{-\lambda}$ vanishes at $z = \infty$. Hence $q(z)z^{-\lambda}$ is a constant, namely 0. It follows that $p_2(x) = 0$ and $h'_{11}(x) = h_{11}(x)$, so the solution $h_{11}(x)$, $h_{21}(x)$ is unique.

The argument just made shows that if there exists any solution which is asymptotic to $s_{11}(x)$, $s_{21}(x)$ in the sector $-\pi/2 \leq \arg(x-\alpha) \leq \pi/2$, this solution is identical with $h_{11}(x)$, $h_{21}(x)$, even if $a > \frac{1}{2}$ (if $a = 1$, $q(z)z^{-\lambda}$ is to be replaced by $q(z)z^{1-\lambda}$); hence in general no such solution exists if $a > \frac{1}{2}$. The only exception is when $q'(x)$ in (134) is identically zero, in which case $h_{11}(x) \sim s_{11}(x)$ in the sector $-\pi < \arg x < \pi$; this occurs in certain reducible equations (see the end of this section).

A similar argument shows that if $a' \leq \frac{1}{2}$ the solution $g_{12}(x)$, $g_{22}(x)$ in which the value of $\lambda'$ defined above is used is the only solution which is represented asymptotically by $s_{12}(x)$, $s_{22}(x)$ in the sector $\pi/2 \leq \arg(x-\alpha) \leq 3\pi/2$. Since $a \leq \frac{1}{2}$ when $-\pi \leq \arg\varrho - (\arg\sigma - \pi/2) < 0$ and $a' \leq \frac{1}{2}$ when

* We shall continue to use $\varphi(\log z)$ in this sense, without implying that the function is the same in each case.

$0 \leq \arg\varrho - (\arg\sigma - \pi/2) < \pi$, it follows that either $h_{11}(x)$, $h_{21}(x)$ or $g_{12}(x)$, $g_{22}(x)$ is unique in every case. (As noted above, when $a = \frac{1}{2}$ or $a' = \frac{1}{2}$ we must take $\alpha$ in the right or left half plane respectively.)

To obtain integral and series solutions of eq. (177), we make the Laplace transformation (138), and find that it is satisfied by the integral

$$\int_a^b t^{x-1}(t-\varrho)^\beta e^{\frac{\sigma}{t}}\,dt,$$

provided $a$ and $b$ are so chosen that

$$\left[t^{x+1}(t-\varrho)^{\beta+1} e^{\frac{\sigma}{t}}\right]_a^b = 0.$$

This expression vanishes at $t = 0$, provided $t \to 0$ in such a way that $R(\sigma/t) \to -\infty$; i. e., if a line is drawn through $t = 0$ perpendicular to the line joining $t = 0$ to $t = \sigma$, $t$ must approach 0 on the side of this line opposite to $\sigma$.* The half plane bounded by this line within which $t$ must remain as it approaches 0 we will call for brevity "the half plane opposite $\sigma$". The expression vanishes at $t = \varrho$ if $R(\beta+1) > 0$, and at $t = \infty$ if $R(x+\beta+2) < 0$.

Taking the limits 0, $\varrho$, we obtain the solution

$$h_1(x) = \int_0^\varrho t^{x-1}(t-\varrho)^\beta e^{\frac{\sigma}{t}}\,dt, \tag{182}$$

where the path of integration starts from $t = 0$ in the half plane opposite $\sigma$; if $\varrho$ lies in this half plane, the path may be taken as a straight line; otherwise we will let it make a partial negative circuit about $t = 0$ to the straight line joining 0 and $\varrho$, and then follow this line to $t = \varrho$. Along the straight line let $\arg t = \arg\varrho$ and $\arg(t-\varrho) = \arg\varrho + \pi$. The integral is valid if $R(\beta+1) > 0$; if $R(\beta+1) \leq 0$, we may use a loop which starts and ends at $t = 0$ in the half plane opposite $\sigma$ and passes around $t = \varrho$ in the positive direction; this gives a solution

* Also the path of approach must not be tangent to the bounding line.

$$h_1'(x) = (1 - e^{2\pi i\beta})\, h_1(x), \tag{183}$$

which we will take as defining $h_1(x)$ in this case. (We exclude temporarily the case where $\beta$ is a negative integer.) The solution $h_1(x)$ is analytic for all finite values of $x$.

Taking the limits $\infty$, $\varrho$, we obtain the solution

$$g_1(x) = \int_\infty^\varrho t^{x-1}(t-\varrho)^\beta\, e^{\frac{\sigma}{t}}\, dt, \tag{184}$$

where the path of integration may be taken as the prolongation of the straight line joining 0 to $\varrho$. We will take $\arg t = \arg(t-\varrho) = \arg \varrho$. The integral is valid if $R(\beta+1) > 0$, $R(x+\beta) < 0$ (cf. footnote, p. 96); if $R(\beta+1) \leq 0$, we may use a loop from $\infty$ which passes around $t = \varrho$ in the positive direction; this gives a solution

$$g_1'(x) = (1 - e^{2\pi i\beta})\, g_1(x), \tag{185}$$

which we will take as defining $g_1(x)$ in this case (again we exclude negative integral values of $\beta$).

Setting $t = \varrho/\tau$ in (184), we have

$$g_1(x) = -\varrho^{x+\beta}\int_0^1 \tau^{-x-\beta-1}(1-\tau)^\beta\, e^{\frac{\sigma}{\varrho}\tau}\, d\tau, \tag{186}$$

or, if we introduce the factor $e^{\frac{\sigma}{\varrho}}$ into the integrand and write $\sigma/\varrho = \varkappa$,

$$g_1(x) = -e^{\varkappa}\varrho^{x+\beta}\int_0^1 \tau^{-x-\beta-1}(1-\tau)^\beta\, e^{-\varkappa(1-\tau)}\, d\tau.$$

Expanding the factor $e^{-\varkappa(1-\tau)}$ and integrating term by term, we have

$$g_1(x) = -e^{\varkappa}\varrho^{x+\beta}\sum_{r=0}^{\infty}\frac{(-\varkappa)^r}{r!}\int_0^1 \tau^{r-x-\beta-1}(1-\tau)^{r+\beta}\, d\tau$$

$$= -e^{\varkappa}\varrho^{x+\beta}\sum_{r=0}^{\infty}\frac{(-\varkappa)^r}{r!}\, B(r-x-\beta,\, r+\beta+1)$$

$$\begin{cases} = -e^{\varkappa}\varrho^{x+\beta}\, B(-x-\beta,\, \beta+1) \\ \quad \cdot \left[1 + \dfrac{(\beta+1)\varkappa}{1\cdot(x-1)} + \dfrac{(\beta+1)(\beta+2)\varkappa^2}{1\cdot 2(x-1)(x-2)} + \cdots\right]; \end{cases} \tag{187}$$

The series in brackets may be regarded as a degenerate hypergeometric series, with one argument lacking; it may be written $\lim\limits_{\alpha\to\infty} F(\alpha, \beta, 1-x, -\varkappa/\alpha)$. It converges for all values of $x$, $\beta$, $\varrho$, and $\sigma$ except $x = 1, 2, 3, \cdots$; at these points the solution is analytic, however, as we see from the preceding beta function series. Accordingly eq. (187) serves to define $g_1(x)$ when $R(x+\beta) \geq 0$; it could be used as well as (185) to define it when $R(\beta+1) < 0$. We see that $g_1(x)$ is analytic throughout the finite part of the plane except for simple poles at $x = -\beta, 1-\beta, 2-\beta, \cdots$. Another form of the series for $g_1(x)$ is obtained if we evaluate (186) without introducing the factor $e^{-\frac{\sigma}{\varrho}}$, namely:

$$g_1(x) = -\varrho^{x+\beta} B(-x-\beta, \beta+1)\left[1 + \frac{(x+\beta)\varkappa}{1\cdot(x-1)} + \frac{(x+\beta)(x+\beta-1)\varkappa^2}{1\cdot 2\,(x-1)(x-2)} + \cdots\right].$$

We can express $g_1(x)$ also by means of a series of partial fractions. Let $c$ be a point on the line joining $\varrho$ to $\infty$, and integrate over the contour consisting of the straight line from $\infty$ to $c$, a loop $k$ about $t = \varrho$, traversed in the positive direction, and the straight line from $c$ back to $\infty$, starting with $\arg t = \arg(t-\varrho) = \arg\varrho$; if $R(x+\beta) < 0$ we have

$$\begin{aligned} g_1'(x) &= (1-e^{2\pi i\beta})\, g_1(x) \\ &= (1-e^{2\pi i\beta})\int_\infty^c t^{x-1} v(t)\, dt + \int_k t^{x-1} v(t)\, dt, \end{aligned}$$

whence if $\beta$ is not an integer

$$g_1(x) = E_1(x) + \int_\infty^c t^{x+\beta-1}\left(1-\frac{\varrho}{t}\right)^{\beta} e^{\frac{\sigma}{t}}\, dt,$$

where $E_1(x)$ is an entire function. Expanding the last two factors of the integrand and integrating term by term, we have

$$g_1(x) = E_1(x) + c^{x+\beta}\left[\frac{1}{x+\beta} + \frac{\sigma - \beta\varrho}{c(x+\beta-1)} + \frac{\sigma^2 - 2\beta\varrho\sigma + \beta(\beta-1)\varrho^2}{2c^2(x+\beta-2)} + \cdots\right].$$

This converges uniformly in the neighborhood of every point except $x = -\beta,\ 1-\beta,\ 2-\beta,\ \cdots$, and hence represents $g_1(x)$ over the whole plane.

If we take the limits $0, \infty$, we get the solution

$$(188)\qquad g_2(x) = \int_0^\infty t^{x-1}(t-\varrho)^\beta e^{\frac{\sigma}{t}}\, dt.$$

The path of integration we may take as any ray in the half plane opposite $\sigma$. We will choose a ray for which $\arg\varrho - 2\pi < \arg t < \arg\varrho$, and take $\arg\varrho < \arg(t-\varrho) < \arg\varrho + 2\pi$. The integral is valid if $R(x+\beta) < 0$.

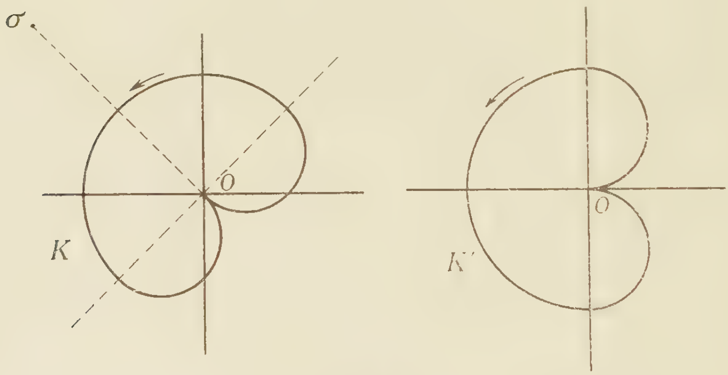


Fig. 13.

A fourth solution is obtained if we take for the path of integration a curve $K$ which starts at $t = 0$ in the half plane opposite $\sigma$, makes a small positive circuit about $t = 0$, and returns to 0 in the same half plane (Fig. 13):

$$(189)\qquad h_2(x) = \int_K t^{x-1}(t-\varrho)^\beta e^{\frac{\sigma}{t}}\, dt.$$

We will let $\arg t$ increase from near $\arg\sigma - \pi$ to near $\arg\sigma + \pi$. If we set $t = -\sigma\tau = e^{\pi i}\sigma\tau$,

$$h_2(x) = (-\varrho)^\beta e^{\pi i x} \sigma^x \int_{K'} \tau^{x-1} (1 + \varkappa\tau)^\beta e^{-\frac{1}{\tau}} d\tau,$$

where $K'$ is the contour in the $\tau$-plane into which $K$ is transformed (Fig. 13; cf. Fig. 4). Let $K$ be small enough so that $|\varkappa\tau| < 1$ at all points of $K'$; then by eq. (72)

$$\begin{aligned} h_2(x) &= (-\varrho)^\beta e^{\pi i x} \sigma^x \int_{K'} \tau^{x-1} \Big[1 + \beta\varkappa\tau \\ &\qquad + \frac{\beta(\beta-1)}{2!}\varkappa^2\tau^2 + \cdots\Big] e^{-\frac{1}{\tau}} d\tau \\ &= (-\varrho)^\beta e^{\pi i x} \sigma^x \Big[-\bar{\Gamma}(-x) - \beta\varkappa\bar{\Gamma}(-x-1) \\ &\qquad - \frac{\beta(\beta-1)}{2!}\varkappa^2 \bar{\Gamma}(-x-2) - \cdots\Big] \\ &= -(-\varrho)^\beta e^{\pi i x} \sigma^x \bar{\Gamma}(-x) \Big[1 - \frac{\beta\varkappa}{1\cdot(x+1)} \\ &\qquad + \frac{\beta(\beta-1)\varkappa^2}{1\cdot 2(x+1)(x+2)} - \cdots\Big], \end{aligned}$$

or, by eq. (60),

$$h_2(x) = 2\pi i(-\varrho)^\beta \frac{\sigma^x}{\Gamma(x+1)} \Big[1 - \frac{\beta\varkappa}{1\cdot(x+1)} + \frac{\beta(\beta-1)\varkappa^2}{1\cdot 2(x+1)(x+2)} - \cdots\Big]. \tag{190}$$

This series converges for all values of $x$, $\beta$, $\varrho$, and $\sigma$ except $x = -1, -2, -3, \cdots$, at which points the solution is, however, analytic; hence $h_2(x)$ is an entire function.

To obtain the relations between these four solutions, let us integrate $t^{x-1}(t-\varrho)^\beta e^{\frac{\sigma}{t}}$ over the contour $ABCDEFGHIJA$ of Fig. 14, made up of straight line segments and circular arcs. The value of the integral is zero, since the contour contains no singular points of the integrand. If $R(\beta) > 0$ and $R(x+\beta) < 0$, we can let the radii of the arcs $BC$ and $GH$ approach 0, and that of the arc $AJI$ increase indefinitely; we can also let the points $D$ and $F$ move into the half plane opposite $\sigma$ and approach the origin, so that

the arc $DEF$ becomes equivalent to the contour $K$ of Fig. 13. Then we have (omitting the integrands)

$$\int_{\infty}^{\varrho} + \int_{\varrho}^{0} + \int_{K} + \int_{0}^{\varrho} + \int_{\varrho}^{\infty} = 0.$$

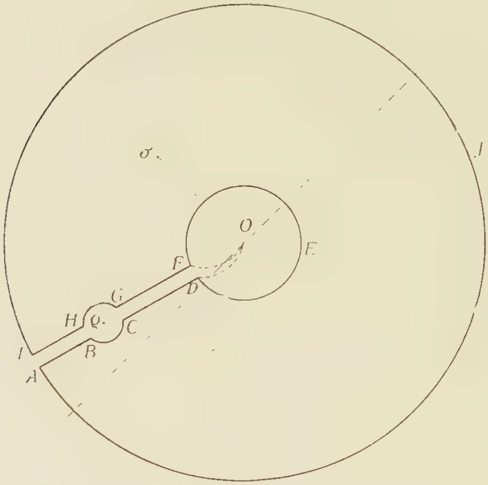


Fig. 14.

On $AB$ take $\arg t = \arg(t-\varrho) = \arg\varrho$; then

$$g_1(x) - h_1(x) + e^{2\pi i x} h_2(x) + e^{2\pi i x} h_1(x) - e^{2\pi i (x+\beta)} g_1(x) = 0,$$

or

$$(1 - e^{2\pi i (x+\beta)}) g_1(x) = (1 - e^{2\pi i x}) h_1(x) - e^{2\pi i x} h_2(x).$$

In Fig. 14 $\arg\varrho > \arg\sigma - \pi/2$, so $\lambda = 1$. If $\arg\varrho < \arg\sigma - \pi/2$, whence $\lambda = 0$, we have similarly, starting with $\arg t = \arg(t-\varrho) = \arg\varrho$ on $AB$ (Fig. 15),

$$g_1(x) - h_1(x) + h_2(x) + e^{2\pi i x} h_1(x) - e^{2\pi i (x+\beta)} g_1(x) = 0,$$

or

$$(1 - e^{2\pi i (x+\beta)}) g_1(x) = (1 - e^{2\pi i x}) h_1(x) - h_2(x).$$

These two results may be combined in the form

$$(191)\quad g_1(x) = \frac{1 - e^{2\pi i x}}{1 - e^{2\pi i (x+\beta)}} h_1(x) - \frac{e^{2\lambda\pi i x}}{1 - e^{2\pi i (x+\beta)}} h_2(x).$$

Now let $\arg \varrho \geqq \arg \sigma - \pi/2$ $(\lambda = 1)$ again, and integrate over the contour $GABCDEFG$ of Fig. 16, takting

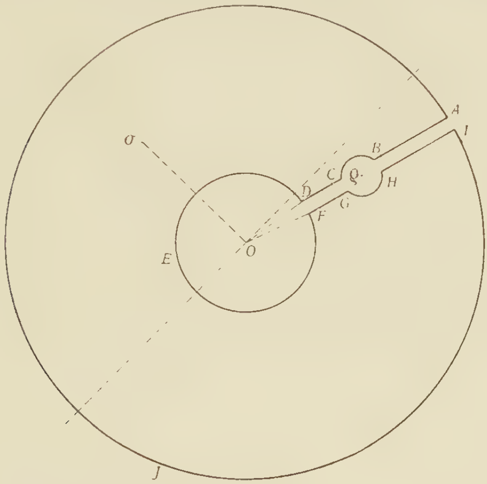


Fig. 15.

$\arg t = \arg \varrho$ and $\arg (t - \varrho) = \arg \varrho + \pi$ on $AB$. If $R(\beta) > 0$ and $R(x + \beta) < 0$ we can let the radius of the arc $BC$ approach 0 and that of $DEF$ increase indefinitely; we may also let $G$ approach 0; then we have

$$h_1(x) - g_1(x) - e^{2\pi i x} g_2(x) = 0,$$

or

$$h_1(x) = g_1(x) + e^{2\pi i x} g_2(x).$$

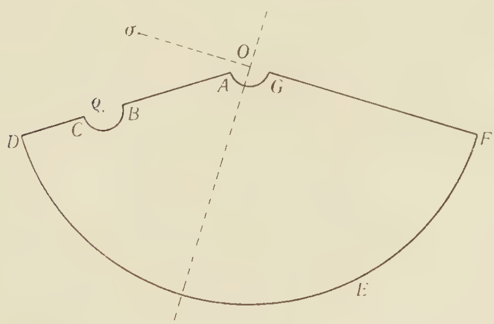


Fig. 16.

If $\arg \varrho < \arg \sigma - \pi/2$ $(\lambda = 0)$, we have similarly from Fig. 17, starting with $\arg t = \arg \varrho$ and $\arg(t - \varrho) = \arg \varrho + \pi$ on $AB$,

$$h_1(x) - e^{2\pi i\beta} g_1(x) - g_2(x) = 0,$$

or

$$h_1(x) = e^{2\pi i\beta} g_1(x) + g_2(x).$$

These two results may be combined in the form

$$h_1(x) = e^{2(1-\lambda)\pi i\beta} g_1(x) + e^{2\lambda\pi i x} g_2(x). \tag{192}$$

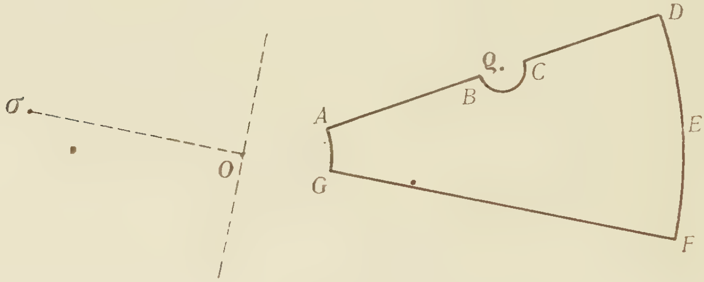


Fig. 17.

Since both sides of eqs. (191) and (192) are analytic, apart from poles, for all values of $x$ and $\beta$, they are identities and we can drop the restrictions placed on $x$ and $\beta$ in deriving them.

By eliminating $h_1(x)$ from eqs. (191) and (192), we find for both $\lambda = 1$ and $\lambda = 0$:

$$h_2(x) = (e^{2\pi i\beta} - 1)\, g_1(x) + (1 - e^{2\pi i x})\, g_2(x). \tag{193}$$

Similarly, eliminating $g_1(x)$ from the same equations, we have

$$g_2(x) = \frac{1 - e^{2\pi i\beta}}{1 - e^{2\pi i(x+\beta)}} h_1(x) + \frac{1}{1 - e^{2\pi i(x+\beta)}} h_2(x)$$

if $\lambda = 1$, and

$$g_2(x) = \frac{1 - e^{2\pi i\beta}}{1 - e^{2\pi i(x+\beta)}} h_1(x) + \frac{e^{2\pi i\beta}}{1 - e^{2\pi i(x+\beta)}} h_2(x).$$

if $\lambda = 0$. These may be combined in the form

$$(194)\quad g_2(x) = \frac{1 - e^{2\pi i\beta}}{1 - e^{2\pi i(x+\beta)}}\, h_1(x) + \frac{e^{2(1-\lambda)\pi i\beta}}{1 - e^{2\pi i(x+\beta)}}\, h_2(x).$$

By means of eqs. (191) and (193) we can express $h_1(x)$ and $g_2(x)$ in terms of $h_2(x)$ and $g_1(x)$, and thus evaluate them in terms of factorial series.

If in eq. (187) we let $x \to \infty$ in the left half plane (whence $-x \to \infty$ in the right half plane), we see that

$$\lim_{x \to \infty} g_1(x)\, \varrho^{-x}\, x^{\beta+1} = e^{\frac{\sigma}{\varrho}} (-\varrho)^{\beta}\, \Gamma(\beta+1);$$

likewise, if in eq. (190) we let $x \to \infty$ in the right half plane,

$$\lim_{x \to \infty} h_2(x)\, x^{x}\, \sigma^{-x}\, e^{-x}\, x^{\frac{1}{2}} = \sqrt{2\pi}\, i\, (-\varrho)^{\beta}.$$

These limits show that $g_1(x) \sim S_1(x)$ in the left half plane and $h_2(x) \sim S_2(x)$ in the right half plane, at least to the first term, if we set

$$(195)\quad s_1 = e^{\frac{\sigma}{\varrho}} (-\varrho)^{\beta}\, \Gamma(\beta+1), \qquad s_2 = \sqrt{2\pi}\, i\, (-\varrho)^{\beta}.$$

It follows that $g_1(x) = g_{11}(x)$ and $h_2(x) = h_{12}(x)$, on account of the uniqueness of these solutions.

Since we know the asymptotic forms of $g_{11}(x)$ and $h_{12}(x)$ in the sectors $0 < \arg x < 2\pi$ and $-\pi < \arg x < \pi$ respectively, we can now use eq. (191) to get the asymptotic form of $h_1(x)$. We have

$$h_1(x) = \frac{1 - e^{2\pi i(x+\beta)}}{1 - e^{2\pi i x}}\, g_1(x) + \frac{e^{2\lambda\pi i x}}{1 - e^{2\pi i x}}\, h_2(x),$$

whence in the upper half plane

$$h_1(x) \sim S_1(x) + e^{2\lambda\pi i x}\, S_2(x).$$

The dominant term depends on the factors $\varrho^{x}$, $e^{2\lambda\pi i x}\, x^{-x}\, \sigma^{x}\, e^{x}$, or on the exponents

$$0,\ (2\lambda\pi i - \log x - \log\varrho + \log\sigma + 1)\, x;$$

the real part of the second exponent is

$$(196)\quad u(-\log|x|+b)-2\pi v\left(\lambda-\frac{\arg x+\arg\varrho-\arg\sigma}{2\pi}\right).$$

This is negative in the sector $0<\arg x<\pi/2$, on account of the term $-u\log|x|$, so the first term dominates and $h_1(x)\sim S_1(x)$ there.

In the lower half plane

$$h_1(x)\sim e^{2\pi i\beta}\,S_1(x)-e^{2(\lambda-1)\pi ix}\,S_2(x),$$

or, if we use the determination of $S_1(x)$ for $-\pi<\arg x<0$,

$$h_1(x)\sim S_1(x)-e^{2(\lambda-1)\pi ix}\,S_2(x).$$

Here the exponents are

$$0,\quad [2(\lambda-1)\pi i-\log x-\log\varrho+\log\sigma+1]\,x,$$

and the real part of the second one is

$$(197)\quad u(-\log|x|+b)-2\pi v\left(\lambda-1-\frac{\arg x+\arg\varrho-\arg\sigma}{2\pi}\right).$$

This is negative in the sector $-\pi/2<\arg x<0$, so $h_1(x)\sim S_1(x)$.

Hence $h_1(x)\sim S_1(x)$ in the right half plane, except possibly along rays near the positive axis of reals, where the asymptotic form of $g_1(x)$ does not hold. To prove that $h_1(x)\sim S_1(x)$ also along these rays, set $t=\varrho\tau$ in (182):

$$\begin{aligned}
h_1(x) &= (-1)^\beta e^{\frac{\sigma}{\varrho}}\varrho^{x+\beta}\int_0^1\tau^{x-1}(1-\tau)^\beta e^{x\frac{1-\tau}{\tau}}\,d\tau\\
&= (-1)^\beta e^{\frac{\sigma}{\varrho}}\varrho^{x+\beta}\left[\int_0^1\tau^{x-1}(1-\tau)^\beta\,d\tau\right.\\
&\qquad\left.+\int_0^1\tau^{x-1}(1-\tau)^\beta\left(e^{x\frac{1-\tau}{\tau}}-1\right)d\tau\right]
\end{aligned}$$

$$(198)\quad\left\{\begin{aligned}
&= (-1)^\beta e^{\frac{\sigma}{\varrho}}\varrho^{x+\beta}\Big[B(x,\beta+1)\\
&\qquad+\left(\int_0^{\frac12}+\int_{\frac12}^1\right)\tau^{x-1}(1-\tau)^\beta\left(e^{x\frac{1-\tau}{\tau}}-1\right)d\tau\Big].
\end{aligned}\right.$$

If we let $x \to \infty$ along a ray parallel to the positive axis of reals, the integral from 0 to $\frac{1}{2}$ approaches 0 exponentially, on account of the factor $\tau^{x-1}$. For the other integral we have

$$\int_{\frac{1}{2}}^{1} \tau^{x-1}(1-\tau)^{\beta}\left(e^{\varkappa\frac{1-\tau}{\tau}}-1\right)d\tau$$
$$= \int_{\frac{1}{2}}^{1} \tau^{x-2}(1-\tau)^{\beta+1}\left[\varkappa + \frac{\varkappa^2}{2!}\left(\frac{1-\tau}{\tau}\right) + \cdots\right]d\tau;$$

the series converges uniformly on the path of integration; let its maximum absolute value on this path be $M$; then

$$\left|\int_{\frac{1}{2}}^{1} \tau^{x-1}(1-\tau)^{\beta}\left(e^{\varkappa\frac{1-\tau}{\tau}}-1\right)d\tau\right|$$
$$< M\int_{\frac{1}{2}}^{1}\left|\tau^{x-2}(1-\tau)^{\beta+1}\right|d\tau$$
$$< M\int_{0}^{1}\tau^{u-2}(1-\tau)^{b+1}\,d\tau = MB(u-1,\, b+2),$$

where $u$ and $b$ are the real parts of $x$ and $\beta$. As $x \to \infty$ this is asymptotic to $M\Gamma(b+2)\,u^{-b-2}$. Since the ratio of $|x^{\beta+1}|$ to $u^{b+1}$ approaches 1 as $x \to \infty$,

$$\lim_{x\to\infty} x^{\beta+1}\int_{\frac{1}{2}}^{1}\tau^{x-1}(1-\tau)^{\beta}\left(e^{\varkappa\frac{1-\tau}{\tau}}-1\right)d\tau = 0.$$

Since $B(x, \beta+1) \sim \Gamma(\beta+1)\,x^{-\beta-1}$, we have from (198)

$$\lim_{x\to\infty} \varrho^{-x}\,x^{\beta+1}\,h_1(x) = (-1)^{\beta}\,e^{\frac{\sigma}{\varrho}}\,\varrho^{\beta}\,\Gamma(\beta+1) = s_1.$$

The asymptotic form of $h_1(x)$ therefore holds, at least to the first term, near the positive axis of reals.

To identify $h_1(x)$ with $h_{11}(x)$, we need to examine its asymptotic form along rays parallel to the axis of imaginaries. Let $x \to \infty$ along any ray parallel to the positive axis of imaginaries; then $\arg x \to \pi/2$ and the expression (196) approaches

$$u(-\log|x| + b) - 2\pi a v,$$

which is ultimately negative, even if $u < 0$, since $a$ and $v$ are positive and $v$ increases more rapidly than $\log |x|$. Now let $x \to \infty$ along any ray parallel to the negative axis of imaginaries: as $\arg x \to -\pi/2$, the expression (197) approaches

$$u(-\log |x| + b) - 2\pi v \left(a - \frac{1}{2}\right);$$

since $v < 0$, this is ultimately negative if $a < \frac{1}{2}$; if $a = \frac{1}{2}$, it is negative if $u > 0$ and positive if $u < 0$. Hence $h_1(x) \sim S_1(x)$ along all rays parallel to the positive axis of imaginaries, and also along all rays parallel to the negative axis of imaginaries if $a < \frac{1}{2}$: it thus has the properties which characterize $h_{11}(x)$.

By a similar procedure we can identify $g_2(x)$ with $g_{12}(x)$. From eq. (193) we have

$$g_2(x) = \frac{1}{1 - e^{2\pi i x}} h_2(x) - \frac{1 - e^{2\pi i \beta}}{1 - e^{2\pi i x}} g_1(x).$$

whence in the upper half plane

$$g_2(x) \sim S_2(x) + (1 - e^{2\pi i \beta}) S_1(x).$$

The dominant term depends on the exponents

$$0, \quad (\log x + \log \varrho - \log \sigma - 1)\, x,$$

the real part of the second of which is

(199) $$u(\log |x| - b) - v(\arg x + \arg \varrho - \arg \sigma).$$

In the lower half plane, if we take $\pi < \arg x < 2\pi$,

$$g_2(x) \sim S_2(x) - (1 - e^{2\pi i \beta})\, e^{-2\pi i x} S_1(x),$$

and the dominant term depends on the exponents

$$0, \quad (\log x + \log \varrho - \log \sigma - 1 - 2\pi i)\, x,$$

the real part of the second of which is

$$(200)\quad u(\log|x| - b) - 2\pi v\left(\frac{\arg x + \arg\varrho - \arg\sigma}{2\pi} - 1\right).$$

From these we see that $g_2(x) \sim S_2(x)$ in both the second and the third quadrant, except possibly near the negative axis of reals, where the asymptotic form of $h_2(x)$ does not hold.

To prove that $g_2(x) \sim S_2(x)$ along rays near the negative axis of reals, set $t = -\sigma\tau = e^{-\pi i}\sigma\tau$ in (188):

$$\begin{aligned} g_2(x) &= (-\varrho)^\beta(-\sigma)^x \int_0^\infty \tau^{x-1}(1+\varkappa\tau)^\beta e^{-\frac{1}{\tau}}\,d\tau \\ &= (-\varrho)^\beta(-\sigma)^x\Big[\int_0^\infty \tau^{x-1} e^{-\frac{1}{\tau}}\,d\tau \\ &\qquad + \int_0^\infty \tau^{x-1}\{(1+\varkappa\tau)^\beta - 1\}\, e^{-\frac{1}{\tau}}\,d\tau\Big] \\ &= (-\varrho)^\beta(-\sigma)^x\Big[\Gamma(-x) + \left(\int_0^2 + \int_2^\infty\right)\tau^{x-1} \\ &\qquad \times\{(1+\varkappa\tau)^\beta - 1\}\, e^{-\frac{1}{\tau}}\,d\tau\Big] \\ &= (-\varrho)^\beta e^{-\pi i x}\sigma^x\Big[\frac{2\pi i e^{\pi i x}}{\Gamma(x+1)} + \int_0^2 \tau^x \frac{(1+\varkappa\tau)^\beta - 1}{\tau}\, e^{-\frac{1}{\tau}}\,d\tau \\ &\qquad + \int_2^\infty \tau^{x-1}\{(1+\varkappa\tau)^\beta - 1\}\, e^{-\frac{1}{\tau}}\,d\tau\Big]. \end{aligned}$$

The function $[(1+\varkappa\tau)^\beta - 1]/\tau$ is analytic between 0 and 2; let $M$ be its maximum absolute value; then

$$\left|\int_0^2 \tau^x \frac{(1+\varkappa\tau)^\beta - 1}{\tau}\, e^{-\frac{1}{\tau}}\,d\tau\right| < M\int_0^2 \left|\tau^x e^{-\frac{1}{\tau}}\right| d\tau$$

$$< M\int_0^\infty \tau^u e^{-\frac{1}{\tau}}\,d\tau = M\Gamma(-u-1) = \frac{-2\pi i e^{\pi i u} M}{(u+1)\,\Gamma(u+1)}.$$

Since the ratio of $|\Gamma(x+1)|$ to $\Gamma(u+1)$ remains finite as $x \to \infty$ along any ray parallel to the negative axis of reals,

$$\lim_{x\to\infty} \bar{\Gamma}(x+1)\int_0^2 \tau^x \frac{(1+\varkappa\tau)^\beta - 1}{\tau} e^{-\frac{1}{\tau}} d\tau = 0.$$

The integral from 2 to $\infty$ approaches 0 exponentially, on account of the factor $\tau^{x-1}$. Hence

$$\lim_{x\to\infty} \sigma^{-x}\bar{\Gamma}(x+1)\, g_2(x) = 2\pi i(-\varrho)^\beta = \sqrt{2\pi}\, s_2,$$

and since $\Gamma(x+1) \sim x^x e^{-x} x^{\frac{1}{2}} \sqrt{2\pi}$, we see that $g_2(x) \sim S_2(x)$, at least to the first term.

To determine the asymptotic form of $g_2(x)$ in the direction of the axis of imaginaries, let $x \to \infty$ along any ray parallel to the positive axis of imaginaries: the expression (199) approaches

$$u(\log|x| - b) - 2\pi v(\tfrac{1}{2} - a');$$

this is ultimately negative if $a' < \frac{1}{2}$, while if $a' = \frac{1}{2}$ it is negative if $u < 0$ and positive if $u > 0$. Now let $x \to \infty$ along any ray parallel to the negative axis of imaginaries; the expression (200) approaches

$$u(\log|x| - b) + 2\pi a' v,$$

which is ultimately negative. Hence in any case $g_2(x) \sim S_2(x)$ in the sector $\pi/2 < \arg(x-\alpha) \leq 3\pi/2$, and if $a' < \frac{1}{2}$ in the sector $\pi/2 \leq \arg(x-\alpha) \leq 3\pi/2$; these properties show that $g_2(x)$ is equal to $g_{12}(x)$.

Since $h_1(x)$, $h_2(x)$, $g_1(x)$, and $g_2(x)$ are the principal solutions of eq. (177), the relations (191)–(194) give us the fundamental periodic functions, whose matrix is therefore

$$P(x) = \begin{Vmatrix} e^{2(1-\lambda)\pi i\beta} & c_1 \\ c_2\, e^{2\lambda\pi i x} & 1 - e^{2\pi i x} \end{Vmatrix},$$

where

(201) $$c_1 = e^{2\pi i\beta} - 1, \qquad c_2 = 1.$$

If $\beta$ is a negative integer, we will define $h_1'(x)$ and $g_1'(x)$ [eqs. (183), (185)] as the principal solutions in place of $h_1(x)$

and $g_1(x)$, and accordingly replace $\Gamma(\beta+1)$ by $\Gamma(\beta+1)$ in $s_1$ [eq. (195)]. If we multiply eqs. (191) and (192) by $1-e^{2\pi i\beta}$ and let $\beta$ approach a negative integer, both reduce to $h_1'(x) = g_1'(x)$.

Since $h_2(x) = h_{12}(x)$, it is represented asymptotically by $S_2(x)$ in the sector $-\pi < \arg x < \pi$. Along a ray parallel to the negative axis of reals

$$h_2(x) \sim [1+\pi(x)]\, S_2(x)$$

as in the general case, $\pi(x)$ being a periodic function such that $|\pi(x)| < M e^{-2\pi|v|}$. A similar statement holds with regard to $g_1(x)$.

The expressions (196) and (197) are ultimately positive along any ray in the second and third quadrants respectively, so $h_1(x) \sim e^{2\lambda\pi i x}\, S_2(x)$ in the second quadrant and $h_1(x) \sim -e^{2(\lambda-1)\pi i x}\, S_2(x)$ in the third quadrant. Similarly (199) and (200) are ultimately positive along any ray in the first and fourth quadrants respectively, so $g_2(x) \sim -c_1\, S_1(x)$ in the first quadrant and $g_2(x) \sim c_1\, e^{-2\pi i x}\, S_1(x)$ in the fourth quadrant.

These solutions do not have "critical rays" like those of the corresponding solutions in the general case; the boundaries between regions in which the asymptotic form is different are not straight lines, but transcendental curves, whose equations are obtained by setting the expressions (196), (197), (199), and (200) equal to zero.

Analytic expressions for the intermediate solutions $y_{11}'(x)$ and $y_{12}'(x)$ can be obtained exactly as in the general case, namely

$$\begin{aligned} y_{11}'(x) &= h_1(x) - \frac{e^{2\lambda\pi i x}}{1-e^{2\pi i x}}\, h_2(x) = \frac{1-e^{2\pi i(x+\beta)}}{1-e^{2\pi i x}}\, g_1(x) \\ &= e^{\varrho}\, (-\varrho)^{\beta}\, \varrho^{x}\, B(x, \beta+1) \left[ 1 + \frac{(\beta+1)\varkappa}{1\cdot(x-1)} \right. \\ &\qquad \left. + \frac{(\beta+1)(\beta+2)\varkappa^2}{1\cdot 2\,(x-1)(x-2)} + \ldots \right]. \end{aligned}$$

$$\bar{y}'_{12}(x) = g_2(x) + \frac{c_1}{1-e^{2\pi i x}} g_1(x) = \frac{1}{1-e^{2\pi i x}} h_2(x)$$

$$= 2\pi i(-\varrho)^\beta \frac{\sigma^x}{\Gamma(x+1)}\left[1 - \frac{\beta \varkappa}{1\cdot(x+1)} + \frac{\beta(\beta-1)\varkappa^2}{1\cdot 2(x+1)(x+2)} + \cdots\right].$$

Both of these are analytic everywhere in the plane except for simple poles at all points congruent to $x=0$; $y'_{11}(x)$ has zeros at $x=-\beta-1$ and points congruent on the left.

Another set of formulas expressing the principal solutions as definite integrals may be obtained as follows. If in eq. (177) we set $y(x) = z(x)/\Gamma(x+\beta+1)$, then $z(x)$ satisfies the equation

(202) $z(x+2) - [\varrho(x+1)+\sigma]z(x+1) + \varrho\sigma(x+\beta+1)z(x) = 0;$

applying the Laplace transformation (138), we find that a solution of this is the integral

$$z(x) = \int_a^b t^{x+\beta}(t-\sigma)^{-\beta-1} e^{-\frac{t}{\varrho}}\,dt,$$

provided $a$ and $b$ are so chosen that

$$\left[t^{x+\beta+1}(t-\sigma)^{-\beta} e^{-\frac{t}{\varrho}}\right]_a^b = 0.$$

This expression vanishes at $t=0$ if $R(x+\beta+1)>0$; at $t=\sigma$ if $R(\beta)<0$; and at $t=\infty$, provided $t\to\infty$ in such a way that $R(t/\varrho)\to+\infty$; i. e., if a line is drawn through $t=0$ perpendicular to the line joining 0 to $\varrho$, $t$ must approach $\infty$ in that half plane bounded by this line in which $t=\varrho$ lies (which we may call "the half plane opposite $-\varrho$").

Taking the limits 0 and $\infty$, we have the solution

$$h'(x) = \int_0^\infty t^{x+\beta}(t-\sigma)^{-\beta-1} e^{-\frac{t}{\varrho}}\,dt;$$

if $\sigma$ lies in the half plane opposite $-\varrho$, we will take for the path of integration any ray in this half plane on which arg $t$

has a value between $\arg\sigma$ and $\arg\varrho + \pi/2$, and let $\arg(t-\sigma)$ have values between $\arg\sigma$ and $\arg\sigma + \pi$; otherwise we may use any ray in the half plane opposite $-\varrho$, in particular the ray through $-\sigma$, on which we will take $\arg t = \arg(t-\sigma) = \arg\sigma - \pi$. The integral is then valid if $R(x+\beta+1) > 0$.

Taking the limits 0 and $\sigma$, we have the solution

$$h''(x) = \int_0^\sigma t^{x+\beta}(t-\sigma)^{-\beta-1} e^{-\frac{t}{\varrho}}\, dt;$$

the path of integration we take as the straight line from 0 to $\sigma$, on which we will let $\arg t = \arg\sigma$ and $\arg(t-\sigma) = \arg\sigma + \pi$. The integral is valid if $R(\beta) < 0$ and $R(x+\beta+1) > 0$. Setting $t = \sigma\tau$, we have

$$h''(x) = e^{-\pi i(\beta+1)} \sigma^x \int_0^1 \tau^{x+\beta}(1-\tau)^{-\beta-1} e^{-\frac{\sigma}{\varrho}\tau}\, d\tau,$$

or, if we introduce the factor $e^{\frac{\sigma}{\varrho}}$ into the integrand and write $\sigma/\varrho = \varkappa$,

$$\begin{aligned}
h''(x) &= e^{-\pi i(\beta+1)} e^{-\frac{\sigma}{\varrho}} \sigma^x \int_0^1 \tau^{x+\beta}(1-\tau)^{-\beta-1} e^{\varkappa(1-\tau)}\, d\tau \\
&= e^{-\pi i(\beta+1)} e^{-\frac{\sigma}{\varrho}} \sigma^x \int_0^1 \tau^{x+\beta}(1-\tau)^{-\beta-1} \\
&\qquad\qquad \times [1 + \varkappa(1-\tau) + \cdots]\, d\tau \\
&= e^{-\pi i(\beta+1)} e^{-\frac{\sigma}{\varrho}} \sigma^x [B(x+\beta+1, -\beta) \\
&\qquad\qquad + \varkappa B(x+\beta+1, 1-\beta) + \cdots] \\
&= e^{-\pi i(\beta+1)} e^{-\frac{\sigma}{\varrho}} \sigma^x \frac{\Gamma(x+\beta+1)\,\Gamma(-\beta)}{\Gamma(x+1)} \\
&\qquad \times \left[1 - \frac{\beta\varkappa}{1\cdot(x+1)} + \frac{\beta(\beta-1)\varkappa^2}{1\cdot 2(x+1)(x+2)} + \cdots\right] \\
&= \frac{e^{-\pi i(\beta+1)} e^{-\frac{\sigma}{\varrho}}\,\Gamma(-\beta)}{2\pi i(-\varrho)^\beta}\, \Gamma(x+\beta+1)\, h_2(x)
\end{aligned}$$

$$\text{(203)} \qquad = \frac{\Gamma(x+\beta+1)}{c_1 s_1} h_2(x).$$

Now take for the path of integration a curve $L$ which starts at $t=\infty$ in the half plane opposite $-\varrho$, makes a large negative circuit about $t=0$, and returns to $\infty$ in the same half plane; this gives the solution

$$g'(x)=\int_L t^{x+\beta}(t-\sigma)^{-\beta-1}e^{-\frac{t}{\varrho}}\,dt,$$

valid for all values of $x$ and $\beta$. Let the arguments of $t$ and $t-\sigma$ decrease from near $\arg\sigma+2\pi$ to near $\arg\sigma$ if $\sigma$ lies in the half plane opposite $-\varrho$, and from near $\arg\sigma+\pi$ to near $\arg\sigma-\pi$ if it does not. Set $t=\varrho/\tau$; then

$$g'(x)=\varrho^x\int_K\tau^{-x-1}(1-\varkappa\tau)^{-\beta-1}e^{-\frac{1}{\tau}}\,d\tau,$$

where $K$ is a curve which starts from $\tau=0$ in the right half plane, makes a small negative circuit about $\tau=0$, and returns in the right half plane (cf. Fig. 4, p. 53). Let $K$ be so small that $|\varkappa\tau|<1$ at all points on it; then

$$\begin{aligned}
g'(x)&=\varrho^x\int_K\tau^{-x-1}\Big[1+(\beta+1)\varkappa\tau\\
&\qquad+\frac{(\beta+1)(\beta+2)}{1\cdot2}\varkappa^2\tau^2+\cdots\Big]e^{-\frac{1}{\tau}}\,d\tau\\
&=\varrho^x\Big[\Gamma(x)+(\beta+1)\varkappa\Gamma(x-1)\\
&\qquad+\frac{(\beta+1)(\beta+2)}{1\cdot2}\varkappa^2\Gamma(x-2)+\cdots\Big]\\
&=\varrho^x\Gamma(x)\Big[1+\frac{(\beta+1)\varkappa}{1\cdot(x-1)}\\
&\qquad+\frac{(\beta+1)(\beta+2)\varkappa^2}{1\cdot2(x-1)(x-2)}+\cdots\Big]\\
&=\frac{1-e^{2\pi i(x+\beta)}}{e^{\frac{\sigma}{\varrho}}(-\varrho)^\beta\,\Gamma(\beta+1)}\Gamma(x+\beta+1)g_1(x)
\end{aligned}$$

$$\text{(204)}\qquad =\frac{\Gamma(x+\beta+1)}{s_1}g_1(x).$$

A fourth solution of eq. (202) is

$$g''(x) = \int_{\infty}^{(\sigma)} t^{x+\beta}(t-\sigma)^{-\beta-1} e^{-\frac{t}{\varrho}}\, dt.$$

where the path of integration may be taken as the prolongation of the straight line joining 0 and $\sigma$ if $\sigma$ lies in the half plane opposite $-\varrho$; otherwise let us rotate this ray about $t = \sigma$ in the negative direction until it extends into the half plane opposite $-\varrho$. In the former case we will take $\arg t = \arg(t-\sigma) = \arg \sigma$, and in the latter case take the values of the arguments which are between $\arg \sigma$ and $\arg \sigma - \pi$. The integral is valid if $R(\beta) < 0$.

To obtain formulas for $h'(x)$ and $g''(x)$ corresponding to (203) and (204), we need to derive the relations between the

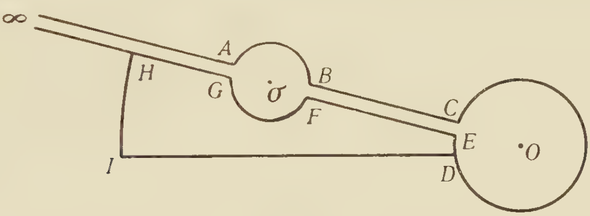


Fig. 18.

solutions of eq. (202). Suppose first that $\arg \varrho \geq \arg \sigma - \pi/2$, or $\lambda = 1$; then $\sigma$ lies in the half plane opposite $-\varrho$. Let us integrate $t^{x+\beta}(t-\sigma)^{-\beta-1} e^{-\frac{t}{\varrho}}$ over the contour $\infty ABCDEFG \infty$ of Fig. 18, which is equivalent to $L$. If $R(\beta+1) < 0$ and $R(x+\beta) > 0$ we can let the radii of the arcs $AB$, $CDE$, and $FG$ approach 0. Let $\arg t = \arg(t-\sigma) = \arg \sigma + 2\pi$ on $\infty A$; then we have

$$g'(x) = e^{2\pi i x} g''(x) - e^{2\pi i(x+\beta)} h''(x) + h''(x) - g''(x),$$

or

$$(205) \qquad g''(x) = \frac{1 - e^{2\pi i(x+\beta)}}{1 - e^{2\pi i x}} h''(x) - \frac{1}{1 - e^{2\pi i x}} g'(x).$$

Integrate now over the contour $EFGHIDE$, taking $\arg t = \arg \sigma$ and $\arg(t-\sigma) = \arg \sigma + \pi$ on $EF$; letting the radii of the

arcs $DE$ and $FG$ approach zero and that of $HI$ increase indefinitely, we have in the limit

$$h''(x)-g''(x)-h'(x) = 0. \tag{206}$$

Putting in eq. (205) the expressions for $h''(x)$ and $g'(x)$ given by (203) and (204), we have

$$g''(x) = \frac{1-e^{2\pi i(x+\beta)}}{1-e^{2\pi ix}} \frac{\Gamma(x+\beta+1)}{c_1 s_1} h_2(x) - \frac{\Gamma(x+\beta+1)}{s_1(1-e^{2\pi ix})} g_1(x)$$
$$- \frac{\Gamma(x+\beta+1)}{c_1 s_1(1-e^{2\pi ix})} [h_2(x)-(e^{2\pi i\beta}-1)g_1(x)],$$

whence, by eq. (193),

$$g''(x) = \frac{\Gamma(x+\beta+1)}{c_1 s_1} g_2(x). \tag{207}$$

Similarly, by using (203) and (207) in eq. (206), we have

$$h'(x) = \frac{\Gamma(x+\beta+1)}{c_1 s_1} h_2(x) - \frac{\Gamma(x+\beta+1)}{c_1 s_1} g_2(x)$$
$$= \frac{\Gamma(x+\beta+1)}{c_1 s_1} [h_2(x)-(1-e^{2\pi i(x+\beta)})g_2(x)],$$

whence, by eq. (194) for $\lambda = 1$,

$$h'(x) = \frac{\Gamma(x+\beta+1)}{s_1} h_1(x). \tag{208}$$

Consider now the case $\arg\varrho < \arg\sigma - \pi/2$, or $\lambda = 0$; $\sigma$ now lies in the half plane opposite $+\varrho$. We can use Fig. 18 again if we interchange the points 0 and $\sigma$. Integrating over the contour $\infty ABCDEFG\infty$, starting with $\arg t = \arg(t-\sigma) = \arg\sigma + \pi$, we have

$$g'(x) = -e^{2\pi ix} h'(x) + h''(x) - e^{2\pi i\beta} h''(x) + h'(x),$$

whence

$$h'(x) = \frac{1}{1-e^{2\pi ix}} [c_1 h''(x) + g'(x)].$$

Integrating over the contour $EFGHIDE$, starting with $\arg t = \arg \sigma$ and $\arg (t-\sigma) = \arg \sigma - \pi$ on $EF$, we have

$$- e^{2\pi i \beta} h''(x) + h'(x) + g''(x) = 0.$$

These two equations lead by the method above to eqs. (207) and (208), which are therefore true for both values of $\lambda$.

From eqs. (203), (204), (207), and (208) we obtain the desired integral expressions for the principal solutions of eq. (177), namely

$$h_1(x) = \frac{s_1}{\Gamma(x+\beta+1)} \int_0^\infty t^{x+\beta} (t-\sigma)^{-\beta-1} e^{-\frac{t}{\varrho}} dt$$
$$[R(x+\beta+1) > 0],$$

$$h_2(x) = \frac{c_1 s_1}{\Gamma(x+\beta+1)} \int_0^\sigma t^{x+\beta} (t-\sigma)^{-\beta-1} e^{-\frac{t}{\varrho}} dt$$
$$[R(\beta) < 0, \quad R(x+\beta+1) > 0],$$

$$g_1(x) = \frac{s_1}{\Gamma(x+\beta+1)} \int_L t^{x+\beta} (t-\sigma)^{-\beta-1} e^{-\frac{t}{\varrho}} dt,$$

$$g_2(x) = \frac{c_1 s_1}{\Gamma(x+\beta+1)} \int_\infty^\sigma t^{x+\beta} (t-\sigma)^{-\beta-1} e^{-\frac{t}{\varrho}} dt,$$
$$[R(\beta) < 0].$$

The restrictions on $x$ and $\beta$ can of course be avoided by replacing the straight line paths by suitable contours.

It follows by the same argument as in the general case that a necessary and sufficient condition for eq. (177) to be reducible is that either $c_1$ or $c_2$ be zero. But $c_2 = 1$, so this is never zero; eq. (177) is reducible, then, if and only if $c_1 = 0$, i. e., if and only if $\beta$ is an integer.

If $\beta$ is zero or a positive integer, $h_2(x)$ and $g_2(x)$ have the respective forms

$$\frac{\sigma^x}{\Gamma(x+\beta+1)} P(x), \quad \frac{\sigma^x}{\Gamma(x+\beta+1)} P(x),$$

where $P(x)$ is a polynomial of degree $\beta$. If $\beta$ is a negative integer, $h_1'(x)$ and $g_1'(x)$ are both equal to $\varrho^x P(x)$, where

$P(x)$ is a polynomial of degree $-\beta-1$; the series $S_1(x)$ contains only $-\beta$ terms in this case, and $h_1'(x) = g_1'(x) = S_1(x)$. Eq. (177) is never completely reducible.

As an example, consider the equation

$$(x+2)\,y(x+2) - xy(x+1) - y(x) = 0,$$

for which $\varrho = 1$, $\sigma = -1$, $\beta = 0$. The formal series have the form

$$S_1(x) = e^{-1}x^{-1}\left(1 - \frac{1}{x} + \frac{0}{x^2} + \cdots\right),$$

$$S_2(x) = \sqrt{2\pi}\, ix^{-x}(-1)^x e^x x^{-\frac{1}{2}}\left(1 - \frac{1}{12x} + \cdots\right).$$

and the principal solutions are

$$h_2(x) = \frac{2\pi i(-1)^x}{\Gamma(x+1)}, \qquad g_2(x) = \frac{2\pi i(-1)^x}{\Gamma(x-1)}.$$

$$h_1(x) = \int_0^1 t^{x-1} e^{-\frac{1}{t}}\, dt = \int_1^\infty \tau^{-x-1} e^{-\tau}\, d\tau = Q(-x),$$

$$g_1(x) = \int_\infty^1 t^{x-1} e^{-\frac{1}{t}}\, dt = -\int_0^1 \tau^{-x-1} e^{-\tau}\, d\tau = -P(-x).$$

where $P(x)$ and $Q(x)$ are Prym's functions (§ 4, Chap. II).

## § 2. Infinite roots.

If one of the roots of the characteristic equation (101) is infinite while the other is finite and different from zero (i. e., $a_2 = 0$, $a_1 \neq 0$, $a_0 \neq 0$), or if both roots are infinite (i. e., $a_2 = a_1 = 0$, $a_0 \neq 0$), the hypergeometric difference equation (99) can readily be transformed into another equation of the same form for which the infinite roots are replaced by zero roots.

Let $a_2 = 0$ in eq. (99), and set $x = -x'$, $y(-x') = f(x')$; then

$$b_2 f(x'-2) + (-a_1 x' + b_1) f(x'-1) + (-a_0 x' + b_0) f(x') = 0;$$

changing $x'$ to $x'+2$, we see that $f(x')$ is a solution of the equation

$$(-a_0 x' - 2a_0 + b_0) f(x'+2) + (-a_1 x' - 2a_1 + b_1) f(x'+1)$$
$$+ b_2 f(x') = 0.$$

This is a hypergeometric difference equation, like eq. (99), but its characteristic equation is

$$-a_0 \varrho^2 - a_1 \varrho = 0,$$

which has one zero root if $a_1 \neq 0$, and two zero roots if $a_1 = 0$. The case of one zero root has already been discussed (§ 1), and that of two zero roots will be taken up in § 4; since $y(x) = f(-x)$ is a solution of the original equation, the two cases named above require no independent treatment.

The equation (96) satisfied by Prym's functions $P_\varrho(x)$ and $Q_\varrho(x)$ is an example of a hypergeometric equation with one root of the characteristic equation infinite. If we make the above transformation it becomes

$$\varrho(x'+2) f(x'+2) - (x' - \varrho + 1) f(x'+1) - f(x') = 0,$$

which for $\varrho = 1$ gives us the numerical example considered at the end of § 1.

Equation (202) also belongs to this type. The method by which this was obtained from eq. (177) suggests an alternative procedure for replacing infinite roots of the characteristic equation by zero roots; if we set

$$y(x) = \Gamma\left(x + \frac{b_0}{a_0}\right) z(x)$$

in eq. (99) (with $a_2 = 0$), it becomes

$$b_2\left(x + \frac{b_0}{a_0} + 1\right) z(x+2) + (a_1 x + b_1) z(x+1) + a_0 z(x) = 0,$$

whose characteristic equation has one zero root if $a_1 \neq 0$ and two if $a_1 = 0$.

It remains to consider the case where one root of the characteristic equation is zero and the other one infinite

(i. e., $a_2 = a_0 = 0,\ a_1 \neq 0$).* To obtain a suitable normal form of eq. (99) for this case, set

$$\frac{b_2}{a_1} = -\frac{1}{\sigma_1}, \quad \frac{b_1}{a_1} = \sigma_3 + 1, \quad \frac{b_0}{a_1} = -\sigma_2;$$

we obtain thus the equation

$$y(x+2) - \sigma_1(x+\sigma_3+1)\,y(x+1) + \sigma_1\sigma_2\,y(x) = 0.$$

Now set $x + \sigma_3 = x'$ and $y(x' - \sigma_3) = f(x')$; dropping the primes, we see that $f(x)$ is a solution of the equation

$$(209) \quad y(x+2) - \sigma_1(x+1)\,y(x+1) + \sigma_1\sigma_2\,y(x) = 0,$$

which we will take as our normal form.

Eq. (209) is satisfied formally by the two power series

$$S_1(x) = x^x \sigma_1^x e^{-x} x^{-\frac{1}{2}}\left(s_1 + \frac{s_1'}{x} + \cdots\right),$$

$$S_2(x) = x^{-x} \sigma_2^x e^{x} x^{-\frac{1}{2}}\left(s_2 + \frac{s_2'}{x} + \cdots\right),$$

where $s_1$ and $s_2$ are arbitrary constants, and

$$\frac{s_1'}{s_1} = \frac{\sigma_2}{\sigma_1} + \frac{1}{12}, \quad \frac{s_2'}{s_2} = -\left(\frac{\sigma_2}{\sigma_1} + \frac{1}{12}\right), \text{ etc.}$$

If we set $y(x) = y_1(x)$, $y(x+1) = y_2(x)$, we obtain the system

$$\begin{cases} y_1(x+1) = y_2(x), \\ y_2(x+1) = -\sigma_1\sigma_2\,y_1(x) + \sigma_1(x+1)\,y_2(x) \end{cases}$$

equivalent to eq. (209); it may be written as a matrix equation

$$Y(x+1) = R(x)\,Y(x),$$

* If $a_2 = a_1 = a_0 = 0$, eq. (99) reduces to one with constant coefficients (§ 4, Chap. I). The same is true if $b_2 = b_1 = b_0 = 0$, but the $a$'s are different from 0.

where

$$R(x) = \begin{Vmatrix} 0 & 1 \\ \sigma_1 \sigma_2 & \sigma_1 (x+1) \end{Vmatrix}.$$

This matrix equation is satisfied formally by

$$(210) \qquad S(x) = \begin{Vmatrix} x^x \sigma_1^x e^{-x} x^{\frac{1}{2}} \left(0 + \frac{s'_{11}}{x} + \cdots\right) & x^{-x} \sigma_2^x e^x x^{-\frac{1}{2}} \left(s_{12} + \frac{s'_{12}}{x} + \cdots\right) \\ x^x \sigma_1^x e^{-x} x^{\frac{1}{2}} \left(s_{21} + \frac{s'_{21}}{x} + \cdots\right) & x^{-x} \sigma_2^x e^x x^{-\frac{1}{2}} \left(0 + \frac{s'_{22}}{x} + \cdots\right) \end{Vmatrix},$$

where $s'_{11} = s_1$, $s_{21} = \sigma_1 s_1$, $s'_{21} = \sigma_1 s'_1$, $s_{12} = s_2$, $s'_{12} = s'_2$, $s'_{22} = \sigma_2 s_2$, etc. The determinant of $S(x)$ has the form

$$|S(x)| = \sigma_1^x \sigma_2^x \left(d + \frac{d'}{x} + \cdots\right),$$

where $d = -\sigma_1 s_1 s_2$. The inverse of $S(x)$ is

$$(211) \quad S^{-1}(x) = \begin{Vmatrix} x^{-x} \sigma_1^{-x} e^x x^{-\frac{1}{2}} \left(0 + \frac{\sigma'_{11}}{x} + \cdots\right) & x^{-x} \sigma_1^{-x} e^x x^{-\frac{1}{2}} \left(\sigma_{12} + \frac{\sigma'_{12}}{x} + \cdots\right) \\ x^x \sigma_2^{-x} e^{-x} x^{\frac{1}{2}} \left(\sigma_{21} + \frac{\sigma'_{21}}{x} + \cdots\right) & x^x \sigma_2^{-x} e^{-x} x^{\frac{1}{2}} \left(0 + \frac{\sigma'_{22}}{x} + \cdots\right) \end{Vmatrix},$$

where $\sigma'_{11} = -\sigma_2/\sigma_1 s_1$, $\sigma_{21} = 1/s_2$, $\sigma_{12} = 1/\sigma_1 s_1$, $\sigma'_{22} = -1/\sigma_1 s_2$, etc.

We can form the matrix products $H_n(x)$ and $G_n(x)$ and prove the first existence theorem exactly as in the general case and the case $\varrho_2 = 0$; the statement of the theorem in § 1 remains true for the present case, except that all the

limit functions are now entire functions, since neither $R(x)$ nor $R^{-1}(x)$ has any finite poles. The limit of the determinants is

$$D(x) = \bar{D}(x) = d\,\sigma_1^x\,\sigma_2^x = -s_1\,s_2\,\sigma_1^{x+1}\,\sigma_2^x,$$

from which we see that $d' = d'' = \cdots = 0$ in the series $|S(x)|$ above. In the details of the proof the $\varrho_1$ and $\varrho_2$ of the general case are replaced by $\sigma_1\,x/e$ and $\sigma_2\,e/x$, but otherwise the changes required are slight. Similarly we can prove the existence of intermediate solutions with the usual properties.

The principal solutions are defined as in § 1. The asymptotic form of $h_{11}(x)$ in the right half plane is given by (129) and (134), and since $s_{11}(x)$ contains a factor $x^x$ and $s_{12}(x)$ a factor $x^{-x}$ the second term is the dominant one, except possibly along a ray parallel to the axis of imaginaries; hence $h_{11}(x) \sim s_{11}(x)$ and $h_{21}(x) \sim s_{21}(x)$ in the sector $-\pi/2 < \arg x < \pi/2$, regardless of the value of the integer $\lambda$. Similarly $g_{12}(x) \sim s_{12}(x)$ and $g_{22}(x) \sim s_{22}(x)$ in the sector $\pi/2 < \arg x < 3\pi/2$, regardless of the value of $\lambda'$. All these solutions are entire functions.

The solutions $h_{12}(x)$, $h_{22}(x)$ and $g_{11}(x)$, $g_{21}(x)$ are unique, by the same argument as in the general case, but not the solutions $h_{11}(x)$, $h_{21}(x)$ and $g_{12}(x)$, $g_{22}(x)$, since $\lambda$ and $\lambda'$ are arbitrary. We proceed to determine as in § 1 what values of these integers give us solutions with properties as nearly as possible the same as those of the principal solutions in the general case.

By (129), the asymptotic form of $h_{11}(x)$ in the upper half plane depends on the factors

$$x^{-x}\,\sigma_2^x\,e^x\,e^{2\lambda\pi i x}, \qquad x^x\,\sigma_1^x\,e^{-x},$$

or on the exponents

$$(-2\log x - \log\sigma_1 + \log\sigma_2 + 2 + 2\,\lambda\,\pi\,i)\,x, \qquad 0;$$

the real part of the first is

$$(-2\log|x| - \log|\sigma_1| + \log|\sigma_2| + 2)\,u$$
$$-2\pi\left(\lambda - \frac{2\arg x + \arg\sigma_1 - \arg\sigma_2}{2\pi}\right)v.$$

Keep $u$ fixed, and let $v \to +\infty$; this ultimately becomes negative (whence $h_{11}(x) \sim s_{11}(x)$) if

$$\lambda > \frac{2 \arg x + \arg \sigma_1 - \arg \sigma_2}{2\pi},$$

even if $u$ is negative; or, since $\arg x \to \pi/2$, if

$$\lambda > \frac{\arg \sigma_1 - \arg \sigma_2}{2\pi} + \frac{1}{2}.$$

If $\lambda$ is less than this value the real part of the first argument ulimately becomes positive.

In the lower half plane $\lambda$ is replaced by $\lambda - 1$; hence if we keep $u$ fixed and let $v \to -\infty$ the real part of the first exponent ultimately becomes negative if

$$\lambda < 1 + \frac{2 \arg x + \arg \sigma_1 - \arg \sigma_2}{2\pi}$$

even if $u$ is negative, or, since $\arg x \to -\pi/2$, if

$$\lambda < \frac{\arg \sigma_1 - \arg \sigma_2}{2\pi} + \frac{1}{2}.$$

If $\lambda$ is greater than this value the real part of the first exponent ultimately becomes positive. Since these inequalities are mutually exclusive, we cannot find any value of $\lambda$ for which $h_{11}(x) \sim s_{11}(x)$ along *all* rays parallel to the axis of imaginaries.

Similarly, $g_{12}(x) \sim s_{12}(x)$ along rays parallel to the positive axis of imaginaries if and only if

$$\lambda' > \frac{\arg \sigma_2 - \arg \sigma_1}{2\pi} - \frac{1}{2},$$

and along rays parallel to the negative axis of imaginaries if and only if

$$\lambda' < \frac{\arg \sigma_2 - \arg \sigma_1}{2\pi} - \frac{1}{2}.$$

Let us define $\lambda$ as the smallest integer which exceeds $(\arg\sigma_1 - \arg\sigma_2)/2\pi + \frac{1}{2}$, and write

$$\text{(212)} \qquad \lambda - \left(\frac{\arg\sigma_1 - \arg\sigma_2}{2\pi} + \frac{1}{2}\right) = a;$$

then $0 < a \leqq 1$. With this value of $\lambda$, $h_{11}(x) \sim s_{11}(x)$ along all rays parallel to the positive axis of imaginaries, but not along those parallel to the negative axis of imaginaries. Following the analogy of the general case rather than the case of one zero root, let us define $\lambda'$ as the smallest integer which exceeds or equals $(\arg\sigma_2 - \arg\sigma_1)/2\pi - \frac{1}{2}$, and write

$$\lambda' - \left(\frac{\arg\sigma_2 - \arg\sigma_1}{2\pi} - \frac{1}{2}\right) = a';$$

then $0 \leqq a' < 1$. With this value of $\lambda'$, $g_{12}(x) \sim s_{12}(x)$ along all rays parallel to the positive axis of imaginaries, but not along those parallel to the negative axis of imaginaries.

By choosing suitable determinations of $\arg\sigma_1$ and $\arg\sigma_2$ we may always have

$$\arg\sigma_1 < \arg\sigma_2 \leqq \arg\sigma_1 + 2\pi;$$

then

$$-\frac{1}{2} \leqq \frac{\arg\sigma_1 - \arg\sigma_2}{2\pi} + \frac{1}{2} < \frac{1}{2},$$

so $\lambda = 1$ and $\lambda' = 0$ if $\arg\sigma_1 \geqq \arg\sigma_2 - \pi$, and $\lambda = 0$ and $\lambda' = 1$ if $\arg\sigma_1 < \arg\sigma_2 - \pi$; in both cases $\lambda' = 1 - \lambda$.

The solutions $h_{11}(x)$, $h_{21}(x)$ and $g_{12}(x)$, $g_{22}(x)$ in which the above values of $\lambda$ and $\lambda'$ are used we will call the principal solutions, although they do not have even the limited degree of uniqueness possessed by the corresponding solutions in § 1.

Since these two solutions cannot be identified by means of their asymptotic properties alone, we will make a direct determination by functiontheoretic methods of the form of the fundamental periodic functions, so that they may be identified by means of their expressions in terms of the other two principal solutions.

In the matrix notation, $H(x) = G(x)P(x)$, or $P(x) = G^{-1}(x)H(x)$, where $H(x)$ and $G(x)$ are the matrices of the principal solutions $h_{ij}(x)$ and $g_{ij}(x)$, and $P(x)$ is the matrix of the fundamental periodic functions. In any period strip in the left half plane the elements of $H(x)$ and $G(x)$ are analytic, and the determinant of $G(x)$ does not vanish, since it is equal to $\bar{D}(x) = -s_1 s_2 \sigma_1^{x+1} \sigma_2^x$; hence the elements of $G^{-1}(x)$ are analytic. The elements of $P(x)$, considered as functions of $z = e^{2\pi i x}$, are therefore single-valued and analytic everywhere in the $z$-plane except possibly at $z = 0$ and $z = \infty$.

In the upper part of the strip $P(x) \sim S^{-1}(x) S(x)$, or, by (210) and (211),

$$P(x) \sim \begin{Vmatrix} 1 & \left(\dfrac{\sigma_2 e^2}{\sigma_1 x^2}\right)^x x^{-1}(0) \\ \left(\dfrac{\sigma_1 x^2}{\sigma_2 e^2}\right)^x x(0) & 1 \end{Vmatrix}.$$

If we write $p_{ij}(x) = q_{ij}(z)$, this means for $p_{11}(x)$ and $p_{22}(x)$

$$\lim_{z \to 0} q_{11}(z) = \lim_{z \to 0} q_{22}(z) = 1.$$

For $p_{12}(x)$ we have

$$\lim_{x \to \infty} p_{12}(x)\, x^{2x} \sigma_1^x \sigma_2^{-x} e^{-2x} x = 0.$$

Write $x^{2x} = (e^{-\frac{\pi i}{2}} x)^{2x} e^{\pi i x}$; the modulus of the first factor behaves like $e^{2u} |x|^{2u}$ at the upper end of the strip, so

$$\lim_{x \to \infty} p_{12}(x)\, e^{\pi i x} \sigma_1^x \sigma_2^{-x} e^{-2x} x \,|x|^{2u} = 0,$$

or, in terms of $z$,

$$\lim_{z \to 0} q_{12}(z)\, z^{\frac{1}{2} + \frac{\arg \sigma_1 - \arg \sigma_2}{2\pi} - \frac{b}{2\pi i}} \varphi(\log z) = 0,$$

where $b = \log|\sigma_2| - \log|\sigma_1| + 2$, and $\varphi(\log z)$ behaves like a power of $\log z$. By (212) this may be written

$$\lim_{z \to 0} q_{12}(z)\, z^{\lambda - 1} z^{1 - a - \frac{b}{2\pi i}} \varphi(\log z) = 0;$$

since $0 < a \geq 1$, it follows that

$$\lim_{z \to 0} q_{12}(z)\, z^{\lambda - 1} = \text{constant},$$

for the remaining factors are not sufficient to neutralize a pole of order as high as the first. By exactly the same argument we have for $p_{21}(x)$

$$\lim_{z \to 0} q_{21}(z)\, z^{-\lambda}\, z^{\frac{a+[\text{illegible}]}{2\pi i}}\, \varphi(\log z) = 0,$$

whence, if $a < 1$,

$$\lim_{z \to 0} q_{21}(z)\, z^{-\lambda} = \text{constant}.$$

From these limits we see that we can write

$$(213) \qquad \begin{cases} q_{11}(z) = 1 + k_1' z + \cdots, \\ q_{22}(z) = 1 + k_2' z + \cdots, \\ q_{12}(z) = z^{1-\lambda}(c_1 + c_1' z + \cdots), \\ q_{21}(z) = z^{\lambda}(c_2 + c_2' z + \cdots), \end{cases}$$

where the $k$'s and $c$'s are constants.

In the lower part of the strip

$$H(x) \sim \begin{vmatrix} e^{2(\lambda-1)\pi i x}\, q'(x)\, s_{12}(x) & s_{12}(x) \\ e^{2(\lambda-1)\pi i x}\, q'(x)\, s_{22}(x) & s_{22}(x) \end{vmatrix}$$

here $s_{12}(x)$ and $s_{22}(x)$ denote the determinations of the series for $\arg x$ near $-\pi/2$; if we take $\arg x$ near $3\pi/2$ they are replaced by $-e^{2\pi i x} s_{12}(x)$ and $-e^{2\pi i x} s_{22}(x)$; using the latter determinations, and using $\lambda'$ instead of $\lambda$ ($\lambda' = 1 - \lambda$), we see that $H(x)$ is represented asymptotically by

$$-e^{2\pi i x}\, x^{-x}\, \sigma_2^x\, e^x\, x^{[\text{illegible}]} \begin{vmatrix} [\text{illegible}] & [\text{illegible}] \\ e^{-2\lambda'\pi i x}\, q'(x)\,(0 + \cdots) & 0 + \cdots \end{vmatrix}$$

The inverse of $G(x)$ is

$$G^{-1}(x) = \frac{1}{G(x)} \begin{vmatrix} [\text{illegible}] & [\text{illegible}] \\ -g_{21}(x) & g_{11}(x) \end{vmatrix}$$

where $G(x) = -s_1 s_2 \sigma_1^{x+1} \sigma_2^{x}$. In the lower part of the strip $G^{-1}(x)$ is represented asymptotically by

$$\frac{\sigma_1^{-x}\sigma_2^{-x}}{-s_1 s_2 \sigma_1}\left\|\begin{matrix} e^{2(\lambda-1)\pi i x}\, q'(x)\, s_{21}(x) & -e^{2(\lambda-1)\pi i x}\, q'(x)\, s_{11}(x) \\ -s_{21}(x) & s_{11}(x) \end{matrix}\right\|;$$

$$x^{x}\sigma_2^{-x}e^{-x}x^{\frac{1}{2}}$$

$$=\left\|\begin{matrix} e^{2(\lambda-1)\pi i x}\, q'(x)\left(-\dfrac{1}{s_2}+\cdots\right) & e^{2(\lambda-1)\pi i x}\, q'(x)(0+\cdots) \\ \dfrac{1}{s_2}+\cdots & 0+\cdots \end{matrix}\right\|.$$

Combining these results, we see that $P(x)$ is represented asymptotically in the lower part of the strip by

$$e^{2\pi i x}\left\|\begin{matrix} e^{-2\pi i x}\, q'(x)\, q'(x)(-1), & e^{2(\lambda-1)\pi i x}\, q'(x)(-1) \\ e^{-2\lambda\pi i x}\, q'(x) & 1 \end{matrix}\right\|.$$

From this we obtain directly the limits

$$\lim_{z\to\infty} q_{11}(z) = \text{constant}, \qquad \lim_{z\to\infty} q_{22}(z)\, z^{-1} = -1,$$

$$\lim_{z\to\infty} q_{12}(z)\, z^{-\lambda} \qquad \lim_{z\to\infty} q_{12}(z)\, z^{\lambda-1} = \text{constant},$$

$$\lim_{z\to\infty} q_{21}(z)\, z^{\lambda-1} \qquad \lim_{z\to\infty} q_{21}(z)\, z^{-\lambda} = \text{constant}.$$

Combining these results with (213), we see that $q_{11}(z) = 1$, $q_{22}(z) = 1 - z$, $q_{12}(z) = c_1 z^{1-\lambda}$, $q_{21}(z) = c_2 z^{\lambda}$, whence

$$P(x) = \left\|\begin{matrix} 1 & c_1 e^{2(1-\lambda)\pi i x} \\ c_2 e^{2\lambda\pi i x} & 1 - e^{2\pi i x} \end{matrix}\right\|. \tag{214}$$

The inverse of this matrix is

$$P^{-1}(x) = \frac{1}{1-(1+c_1 c_2)e^{2\pi i x}}\left\|\begin{matrix} 1 - e^{2\pi i x} & -c_1 e^{2(1-\lambda)\pi i x} \\ -c_2 e^{2\lambda\pi i x} & 1 \end{matrix}\right\|;$$

but the elements of $P^{-1}(x)$ can have no poles, since $G(x) = H(x) P^{-1}(x)$ and the elements of $G(x)$ are entire functions: hence $1 + c_1 c_2 = 0$ and

$$(215) \qquad P^{-1}(x) = \begin{Vmatrix} 1 - e^{2\pi i x} & -c_1 e^{2(1-\lambda)\pi i x} \\ -c_2 e^{2\lambda\pi i x} & 1 \end{Vmatrix}.$$

The values of the constants $c_1$ and $c_2$ will be determined later (p. 170).

Since the elements of $H(x)$ and $G(x)$ are analytic functions of $\sigma_1$ and $\sigma_2$ as well as of $x$, the identity $H(x) = G(x) P(x)$ continues to hold in the limiting case where $a = 1$ (i. e., $\arg \sigma_1 = \arg \sigma_2 - \pi$).

Applying the Laplace transformation (138), we find that eq. (209) is satisfied by the integral

$$\int_a^b t^{x-1} e^{-\frac{t}{\sigma_1}} e^{\frac{\sigma_2}{t}} \, dt,$$

provided $a$ and $b$ are so chosen that

$$\left[ t^{x-1} e^{-\frac{t}{\sigma_1}} e^{\frac{\sigma_2}{t}} \right]_a^b = 0.$$

This expression vanishes at $t = 0$, provided $t \to 0$ in the half plane opposite $\sigma_2$, and at $t = \infty$, provided $t \to \infty$ in the half plane opposite $-\sigma_1$.

Let us take for the path of integration a curve $K$ which starts at $t = 0$ in the half plane opposite $\sigma_2$, makes a positive circuit about $t = 0$, and returns to 0 in the same half plane (cf. Fig. 19, p. 171); we obtain thus the solution

$$h_2(x) = \int_K t^{x-1} e^{-\frac{t}{\sigma_1}} e^{\frac{\sigma_2}{t}} \, dt;$$

we will let $\arg t$ have values between $\arg \sigma_2 - \pi$ and $\arg \sigma_2 + \pi$. If we set $t = -\sigma_2 \tau = e^{\pi i} \sigma_2 \tau$,

$$h_2(x) = e^{\pi i x} \sigma_2^x \int_{K'} \tau^{x-1} e^{\varkappa \tau} e^{-\frac{1}{\tau}} \, d\tau.$$

where $\varkappa = \sigma_2/\sigma_1$, and the curve $K'$ is similar to $K$, but starts and ends at $\tau = 0$ in the right half plane; $\arg\tau$ has values between $-2\pi$ and 0. Expanding $e^{\varkappa\tau}$, we have

$$\begin{aligned}
h_2(x) &= e^{\pi i x}\sigma_2^x \int_{K'} \tau^{x-1}\left(1 + \varkappa\tau + \frac{\varkappa^2\tau^2}{2!} + \cdots\right) e^{-\frac{1}{\tau}}\, d\tau \\
&= e^{\pi i x}\sigma_2^x\left[-\bar{\Gamma}(-x) - \varkappa\bar{\Gamma}(-1-x) - \frac{\varkappa^2}{2!}\bar{\Gamma}(-2-x) - \cdots\right] \\
&= -e^{\pi i x}\sigma_2^x\,\bar{\Gamma}(-x)\left[1 - \frac{\varkappa}{x+1} + \frac{\varkappa^2}{1\cdot 2(x+1)(x+2)} - \cdots\right]
\end{aligned}$$

$$(216) \quad = \frac{2\pi i\,\sigma_2^x}{\Gamma(x+1)}\left[1 - \frac{\varkappa}{x+1} + \frac{\varkappa^2}{1\cdot 2(x+1)(x+2)} - \cdots\right].$$

This factorial series converges for all values of $x$, $\sigma_1$, and $\sigma_2$ except $x = -1, -2, -3, \cdots$, at which points the solution is, however, analytic.

Let us now take for the path of integration a curve $L$ which starts at $t = \infty$ in the half plane opposite $-\sigma_1$, makes a positive circuit about $t = \infty$ (i. e., a negative circuit about $t = 0$), and returns to $t = \infty$ in the same half plane, enclosing some ray $\arg t = \theta$, where $\arg\sigma_1 - \pi < \theta < \arg\sigma_1 + \pi$; we may in particular take $\theta = \arg\sigma_1$ (cf. Fig. 19); let $\arg t$ have values between $\theta + 2\pi$ and $\theta$. We obtain thus the solution

$$g_1(x) = \int_L t^{x-1} e^{-\frac{t}{\sigma_1}} e^{\frac{\sigma_2}{t}}\, dt.$$

If we set $t = \sigma_1\tau$,

$$g_1(x) = \sigma_1^x \int_{L'} \tau^{x-1} e^{-\tau} e^{\frac{\varkappa}{\tau}}\, d\tau,$$

where $L'$ is similar to $L$, but encloses some line in the right half plane (the axis of reals if $\theta = \arg\sigma_1$). Expanding $e^{\frac{\varkappa}{\tau}}$ and integrating, we have by eq. (70)

$$g_1(x) = \sigma_1^x\left[\bar{\Gamma}(x) + \varkappa\bar{\Gamma}(x-1) + \frac{\varkappa^2}{2!}\bar{\Gamma}(x-2) + \cdots\right]$$

$$(217) \qquad = \sigma_1^x\,\bar{\Gamma}(x)\left[1 + \frac{\varkappa}{x-1} + \frac{\varkappa^2}{1\cdot 2(x-1)(x-2)} + \cdots\right].$$

This series converges for all values of $x$, $\sigma_1$, and $\sigma_2$ except $x = 1, 2, 3, \cdots$, and even at these points the solution is analytic.

If in eq. (216) we let $x \to \infty$ in the right half plane, we see that

$$h_2(x) \sim 2\pi i \sigma_2^x x^{-x} e^x x^{-\frac{1}{2}} (2\pi)^{\frac{1}{2}} (1 + \cdots);$$

if in eq. (217) we let $x \to \infty$ in the left half plane,

$$g_1(x) \sim \sigma_1^x x^x e^{-x} x^{-\frac{1}{2}} \sqrt{2\pi}\,(1 + \cdots);$$

hence $h_2(x) \sim S_2(x)$ in the right half plane and $g_1(x) \sim S_1(x)$ in the left half plane if we set

$$s_1 = \sqrt{2\pi}, \qquad s_2 = \sqrt{2\pi}\, i.$$

It follows that $h_2(x) = h_{12}(x)$ and $g_1(x) = g_{11}(x)$, on account of the uniqueness of these solutions.

We are now in a position to determine the values of the constants $c_1$ and $c_2$ in the fundamental periodic functions (214) and (215). Expressing $h_{12}(x)$ in terms of $g_{11}(x)$ and $g_{12}(x)$, we have

$$h_{12}(x) = c_1 e^{2(1-\lambda)\pi i x} g_{11}(x) + (1 - e^{2\pi i x})\, g_{12}(x);$$

setting $x = 0$ in this equation, we see that $h_{12}(0) = c_1 g_{11}(0)$, whence

$$c_1 = \frac{h_{12}(0)}{g_{11}(0)} = \frac{h_2(0)}{g_1(0)} = \frac{2\pi i(1 - \varkappa + \cdots)}{-2\pi i(1 - \varkappa + \cdots)} = -1.$$

It follows from the relation $1 + c_1 c_2 = 0$ that $c_2 = 1$.

Using these values in (214) and (215), we have the relations

$$(218)\qquad \begin{cases} h_{11}(x) = g_{11}(x) + e^{2\lambda\pi i x} g_{12}(x), \\ h_{12}(x) = -e^{2(1-\lambda)\pi i x} g_{11}(x) + (1 - e^{2\pi i x})\, g_{12}(x); \end{cases}$$

$$(219)\qquad \begin{cases} g_{11}(x) = (1 - e^{2\pi i x})\, h_{11}(x) - e^{2\lambda\pi i x} h_{12}(x), \\ g_{12}(x) = e^{2(1-\lambda)\pi i x} h_{11}(x) + h_{12}(x). \end{cases}$$

Consider now the solution

$$m(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t}{\sigma_1}} e^{\frac{\sigma_2}{t}} \, dt,$$

where the path of integration is any straight line in the sector common to the half planes opposite $\sigma_2$ and $-\sigma_1$;* i. e., the sector $\arg \sigma_1 - \pi/2 < \arg t < \arg \sigma_2 - \pi/2$ if $\lambda = 1$, and $\arg \sigma_2 - 3\pi/2 < \arg t < \arg \sigma_1 + \pi/2$ if $\lambda = 0$. Consider first the case $\lambda = 1$ (Fig. 19). The contour $L$ can obviously be deformed so that it consists of the straight line from $\infty$ to 0, the curve $K$, traversed in the negative direction, and the straight line from 0 back to $\infty$. Hence

$$g_1(x) = -e^{2\pi i x} m(x) - h_2(x) + m(x),$$

or

$$(220) \quad (1 - e^{2\pi i x}) \, m(x) = g_1(x) + h_2(x) = g_{11}(x) + h_{12}(x).$$

Comparison of this with the second equation in (218) for $\lambda = 1$ shows that $m(x) = g_{12}(x)$.

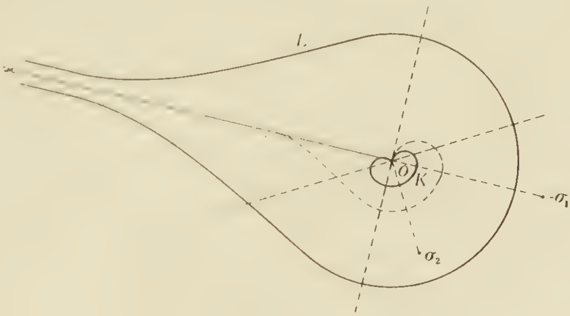


Fig. 19.

Instead of taking the path of integration in $m(x)$ as the straight line from 0 to $\infty$, suppose we start from $t = 0$ in the half plane opposite $\sigma_2$ with $\arg t$ near $\arg \sigma_2 + \pi$, make

* Slight modification of the path is required if $\arg \sigma_2 = \arg \sigma_1 + 2\pi$, so that the two half planes do not overlap.

a negative circuit about 0, and then follow the straight line to $\infty$ (see the dotted curve in Fig. 19). Calling the integral along this path $m'(x)$, we see that

$$m'(x) = -h_2(x) + m(x) = g_{12}(x) - h_{12}(x),$$

and comparison with the second equation in (219) for $\lambda = 1$ shows that $m'(x) = h_{11}(x)$.

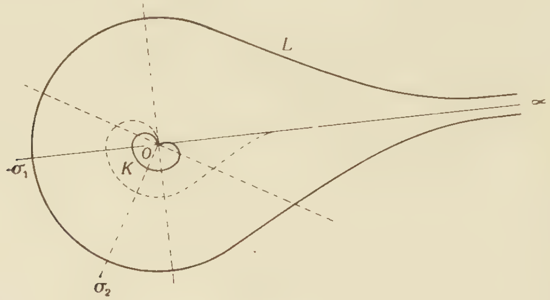


Fig. 20.

In the case $\lambda = 0$ (Fig. 20) the contour $L$ can be modified as before, and eq. (220) still holds. Comparison with the first equation in (219) for $\lambda = 0$ shows that $m(x) = h_{11}(x)$. If in $m(x)$ we take for the path of integration a curve which starts from $t = 0$ in the half plane opposite $\sigma_2$ with arg $t$ near arg $\sigma_2 - \pi$, makes a positive circuit about 0, and then follows the straight line to $\infty$ (see the dotted curve in Fig. 20), we obtain the solution

$$m''(x) = h_2(x) + e^{2\pi i x} m(x) = h_{12}(x) + e^{2\pi i x} h_{11}(x),$$

and comparison with the second equation in (219) for $\lambda = 0$ shows that $m''(x) = g_{12}(x)$.

These results give us definite integral expressions for all the principal solutions. In accordance with our previous usage, we will use the notation $h_1(x)$ and $g_2(x)$ for $h_{11}(x)$ and $g_{12}(x)$ respectively.

By means of eqs. (218) and (219) we can study the asymptotic forms of the principal solutions in the entire plane. We find by the usual methods that

$$h_1(x) \sim \begin{cases} -e^{2(\lambda-1)\pi i x} S_2(x), & -\pi < \arg x \leq -\pi/2; \\ S_1(x), & -\pi/2 < \arg x \leq \pi/2; \\ e^{2\lambda\pi i x} S_2(x), & \pi/2 < \arg x < \pi; \end{cases}$$

$$g_2(x) \sim \begin{cases} e^{2(1-\lambda)\pi i x} S_1(x), & 0 < \arg x < \pi/2; \\ S_2(x), & \pi/2 \leq \arg x < 3\pi/2; \\ -e^{-2\lambda\pi i x} S_1(x), & 3\pi/2 \leq \arg x < 2\pi. \end{cases}$$

As in the case of one zero root (§ 1), the boundaries between the regions where the asymptotic forms are different are not straight lines, but transcendental curves.

Analytic expressions for the intermediate solutions $g'_{11}(x)$ and $\bar{g}'_{12}(x)$ can be obtained exactly as in the general case, namely

$$g'_{11}(x) = h_1(x) - \frac{e^{2\lambda\pi i x}}{1 - e^{2\pi i x}} h_2(x) = \frac{1}{1 - e^{2\pi i x}} g_1(x)$$

$$= \sigma_1^x \Gamma(x) \left[ 1 + \frac{\varkappa}{x-1} + \frac{\varkappa^2}{1 \cdot 2 (x-1)(x-2)} + \cdots \right].$$

$$g'_{12}(x) = g_2(x) - \frac{e^{2(1-\lambda)\pi i x}}{1 - e^{2\pi i x}} g_1(x) = \frac{1}{1 - e^{2\pi i x}} h_2(x)$$

$$= \frac{2\pi i \sigma_2^x}{\Gamma(x+1)} \left[ 1 - \frac{\varkappa}{x+1} + \frac{\varkappa^2}{1 \cdot 2 (x+1)(x+2)} - \cdots \right].$$

These are analytic everywhere except for simple poles at all points congruent to $x = 0$.

If eq. (209) is reducible, it must have a pair of solutions which are represented asymptotically by either $S_1(x)$ or $S_2(x)$, one in the sector $0 < \arg x < 2\pi$ and the other in the sector $-\pi < \arg x < \pi$, and whose ratio is a periodic function of the form (75). Let the series be $S_2(x)$; then one of the solutions must be $h_2(x)$; let $\gamma(x)$ denote the other, which is asymptotic to $S_2(x)$ in the sector $0 < \arg x < 2\pi$. Consider the periodic function $p(x) = \gamma(x)/h_2(x)$; in a period strip sufficiently far to the left $\gamma(x)$ does not vanish, since it is asymptotic to $S_2(x)$, and $h_2(x)$ has no poles, so $p(x)$ does

not vanish; its numerator is therefore 1, i. e., $m = 0$ in (75). The denominator consequently has only one factor, since $m - n = \mu$, and comparison of $S_2(x)$ with (44) shows that $\mu = -1$. Along a ray parallel to the negative axis of imaginaries, if we take arg $x$ near $3\pi/2$ we have $\gamma(x) \sim S_2(x)$ and $h_2(x) \sim - e^{2\pi ix} S_2(x)$; hence $\lim\limits_{x\to\infty} e^{2\pi ix} p(x) = -1$, so $\beta_1 = 0$ and

$$p(x) = \frac{1}{1 - e^{2\pi ix}}, \quad \gamma(x) = \frac{h_2(x)}{1 - e^{2\pi ix}}.$$

The denominator of $\gamma(x)$ vanishes when $x$ is an integer; since $\gamma(x) \sim S_2(x)$ along the negative axis of reals, the numerator must also vanish at these points; but this is impossible, for if any solution of eq. (209) vanishes at two consecutive integers, we see directly from the equation that it must then vanish at all integers, and this is contradicted in the case of $h_2(x)$ by the fact that $h_2(x) \sim S_2(x)$ along the positive axis of reals. It follows that eq. (209) can have no solution which is asymptotic to $S_2(x)$ in the sector $0 < \arg x < 2\pi$. A similar argument shows that eq. (209) can have no solution which is asymptotic to $S_1(x)$ in the sector $-\pi < \arg x < \pi$. Hence eq. (209) is never reducible. (The only exceptions to this statement are the trivial cases $\sigma_2 = 0$ and $\sigma_1 = \infty$, in which eq. (209) reduces to one of the first order.)

The difference equation satisfied by Bessel's function

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k! \Gamma(n+k+1)},$$

regarded as a function of its order $n$, is of the type considered in this section; it has the form

$$y(n+2) - \frac{2}{x}(n+1)\, y(n+1) + y(n) = 0,$$

where $n$ is the independent variable. Comparison with (209) shows that this is in the normal form with $\sigma_1 = 2/x$, $\sigma_2 = x/2$. It has the solutions

$$h_2(n) = 2\pi i J_n(x), \quad g_1(n) = -2\pi i e^{\pi in} J_{-n}(x).$$

## § 3. The case $\varrho_1 = \varrho_2$.

If in eq. (99) $a_2$, $a_1$, and $a_0$ are all different from zero, but $a_1^2 - 4a_0a_2 = 0$, the two roots of the characteristic equation (101) are equal, but finite and different from zero. Let $\varrho$ denote the common value of $\varrho_1$ and $\varrho_2$, namely $\varrho = -a_1/2a_2$, and make the substitution

$$\frac{b_2}{a_2} = \beta + \gamma + 1, \qquad \frac{b_1}{a_2} = -\varrho(\alpha + \beta + 2\gamma + 1),$$
$$\frac{b_0}{a_2} = \varrho^2\gamma;$$

eq. (99) takes the form

$$\begin{aligned}(x + \beta + \gamma + 1)\, y(x+2) \\ - \varrho(2x + \alpha + \beta + 2\gamma + 1)\, y(x+1) \\ + \varrho^2(x+\gamma)\, y(x) = 0;\end{aligned}$$

if we set $x + \gamma = x'$ and $y(x' - \gamma) = f(x')$, then $f(x)$ satisfies the equation

$$\begin{aligned}(x + \beta + 1)\, y(x+2) - \varrho(2x + \alpha + \beta + 1)\, y(x+1) \\ + \varrho^2 x y(x) = 0,\end{aligned} \tag{221}$$

which we will take as our normal form for this case.

If we substitute in eq. (221) a series in powers of $1/x$ such as we have used in the previous cases, we find that it is in general impossible so to determine the constants that the equation is satisfied.* We are led therefore to try series of a different form; analogy with the anormal series for differential equations suggests the trial of a series in powers of $1/\sqrt{x}$, so we will set

$$y(x) = a^x e^{b\sqrt{x}} x^c \left(s + \frac{s'}{\sqrt{x}} + \frac{s''}{x} + \frac{s'''}{x^{3/2}} + \cdots\right).$$

* The case $\alpha = 0$ is an exception; see the end of this section.

The expression $e^{b(\sqrt{x+2}-\sqrt{x})}$ may be expanded as follows:

$$e^{b(\sqrt{x+2}-\sqrt{x})} = e^{b\sqrt{x}\left[\left(1+\frac{2}{x}\right)^{1/2}-1\right]} = e^{\frac{b}{\sqrt{x}}}\, e^{-\frac{b}{2x^{3/2}}}\, e^{\frac{b}{2x^{5/2}}}\cdots$$

$$= \left(1+\frac{b}{\sqrt{x}}+\frac{b^2}{2x}+\frac{b^3}{6x^{3/2}}+\frac{b^4}{24x^2}+\cdots\right)$$

$$\times\left(1-\frac{b}{2x^{3/2}}+\cdots\right)\cdots$$

$$= 1+\frac{b}{\sqrt{x}}+\frac{b^2}{2x}+\frac{1}{x^{3/2}}\left(\frac{b^3}{6}-\frac{b}{2}\right)$$

$$+\frac{1}{x^2}\left(\frac{b^4}{24}-\frac{b^2}{2}\right)+\cdots,$$

and similarly with $e^{b(\sqrt{x+1}-\sqrt{x})}$; otherwise no new difficulties appear, and we find that eq. (221) is satisfied formally by the two series

$$S_1(x) = \varrho^x\, e^{2\sqrt{\alpha}\sqrt{x}}\, x^{-\frac{\beta}{2}-\frac{1}{4}}\left(s_1+\frac{s_1'}{\sqrt{x}}+\frac{s_1''}{x}+\cdots\right).$$

$$S_2(x) = \varrho^x\, e^{-2\sqrt{\alpha}\sqrt{x}}\, x^{-\frac{\beta}{2}-\frac{1}{4}}\left(s_2+\frac{s_2'}{\sqrt{x}}+\frac{s_2''}{x}+\cdots\right).$$

where

$$\frac{s_1'}{s_1} = \frac{4\alpha^2+24\alpha\beta-12\beta^2-24\alpha+3}{48\sqrt{\alpha}}, \qquad \frac{s_2'}{s_2} = -\frac{s_1'}{s_1}, \text{ etc.}$$

The series are uniquely determined, except for the constant factors $s_1$ and $s_2$.

Eq. (221) is unaltered when $\sqrt{x}$ is replaced by $-\sqrt{x}$, so $S_1(x)$ must remain a formal solution after this change: and since the exponential factors go over into those of $S_2(x)$, the whole series must become identical with $S_2(x)$ (apart from a constant factor), on account of the uniqueness of the latter. This shows that

$$\frac{s_2^{(k)}}{s_2} = (-1)^k\,\frac{s_1^{(k)}}{s_1}.$$

If we write the series in the form

$$S'(x) = \varrho^x e^{2\sqrt{\alpha}\sqrt{x}} (\sqrt{x})^{-\beta-\frac{1}{2}} \left(s + \frac{s'}{\sqrt{x}} + \frac{s''}{x} + \cdots\right),$$

$$S''(x) = \varrho^x e^{-2\sqrt{\alpha}\sqrt{x}} (-\sqrt{x})^{-\beta-\frac{1}{2}} \left(s - \frac{s'}{\sqrt{x}} + \frac{s''}{x} - \cdots\right)$$

(taking $s$ the same in both), then $S'(x)$ and $S''(x)$ are interchanged when $x$ makes a circuit about the origin. Thus they represent two determinations of a single series, rather than two distinct series.

The two series are also interchanged if $\sqrt{\alpha}$ is replaced by $-\sqrt{\alpha}$. The ambiguity due to this can be avoided by choosing a definite determination for $\sqrt{\alpha}$. We may assume without loss of generality that $\alpha$ is in the upper half plane, since if we set $x = -x' - \beta - 1$ and $y(-x' - \beta + 1) = \varrho^{-2x'} f(x')$ in eq. (221) we obtain a new equation with exactly the same form except that $\alpha$ is replaced by $-\alpha$. Accordingly we will take $0 \leq \arg \alpha < \pi$, and $\arg \sqrt{\alpha} = \frac{1}{2} \arg \alpha$. The coefficients in the series above contain $\alpha$ in their denominators, so we will assume for the present that $\alpha \neq 0$.

If we set $y(x) = y_1(x)$, $y(x+1) = y_2(x)$, eq. (221) is replaced by the system

$$(222) \quad \begin{cases} y_1(x+1) = y_2(x), \\ y_2(x+1) = -\dfrac{\varrho^2 x}{x+\beta+1} y_1(x) \\ \qquad\qquad + \dfrac{\varrho(2x+\alpha+\beta+1)}{x+\beta+1} y_2(x), \end{cases}$$

which may be written as a matrix equation

$$(223) \qquad Y(x+1) = R(x)\, Y(x),$$

where

$$R(x) = \left\| \begin{matrix} 0 & 1 \\ -\dfrac{\varrho^2 x}{x+\beta+1} & \dfrac{\varrho(2x+\alpha+\beta+1)}{x+\beta+1} \end{matrix} \right\|.$$

Eq. (223) is satisfied formally by the matrix of series

$$S(x) = \varrho^x x^{-\frac{\beta}{2}-\frac{1}{4}}$$

$$\times \left\| \begin{matrix} e^{2\sqrt{\alpha}\sqrt{x}}\left(s_{11}+\dfrac{s'_{11}}{\sqrt{x}}+\cdots\right) & e^{-2\sqrt{\alpha}\sqrt{x}}\left(s_{12}+\dfrac{s'_{12}}{\sqrt{x}}+\cdots\right) \\ e^{2\sqrt{\alpha}\sqrt{x}}\left(s_{21}+\dfrac{s'_{21}}{\sqrt{x}}+\cdots\right) & e^{-2\sqrt{\alpha}\sqrt{x}}\left(s_{22}+\dfrac{s'_{22}}{\sqrt{x}}+\cdots\right) \end{matrix} \right\|,$$

where $s_{11} = s_1$, $s'_{11} = s'_1$, $s_{12} = s_2$, $s'_{12} = s'_2$, $s_{21} = \varrho s_1$, $s'_{21} = \varrho(s'_1 + \sqrt{\alpha}\, s_1)$, $s_{22} = \varrho s_2$, $s'_{22} = \varrho(s'_2 - \sqrt{\alpha}\, s_2)$, etc. The determinant of $S(x)$ has the form

$$|S(x)| = \varrho^{2x} x^{-\beta-1}\left(d + \frac{d'}{x} + \frac{d''}{x^2} + \cdots\right),$$

where $d = -2\sqrt{\alpha}\, \varrho s_1 s_2$, etc. The inverse of $S(x)$ is

$$S^{-1}(x) = \varrho^{-x} x^{\frac{\beta}{2}+\frac{3}{4}}$$

$$\times \left\| \begin{matrix} e^{-2\sqrt{\alpha}\sqrt{x}}\left(\sigma_{11}+\dfrac{\sigma'_{11}}{\sqrt{x}}+\cdots\right) & e^{-2\sqrt{\alpha}\sqrt{x}}\left(\sigma_{12}+\dfrac{\sigma'_{12}}{\sqrt{x}}+\cdots\right) \\ e^{2\sqrt{\alpha}\sqrt{x}}\left(\sigma_{21}+\dfrac{\sigma'_{21}}{\sqrt{x}}+\cdots\right) & e^{2\sqrt{\alpha}\sqrt{x}}\left(\sigma_{22}+\dfrac{\sigma'_{22}}{\sqrt{x}}+\cdots\right) \end{matrix} \right\|,$$

where $\sigma_{11} = -1/2\sqrt{\alpha}\, s_1$, $\sigma_{12} = 1/2\sqrt{\alpha}\, \varrho s_1$, $\sigma_{21} = 1/2\sqrt{\alpha}\, s_2$, $\sigma_{22} = -1/2\sqrt{\alpha}\, \varrho s_2$, etc.

Eq. (223) is satisfied formally by the infinite products (113), and by forming the products $H_n(x)$ and $G_n(x)$ as in the general case and letting $n \to \infty$ we can obtain two analytic solutions of the system (222). The first part of the proof requires only slight modification from that in § 4. Chap. III; thus eq. (115) is replaced by

$$T^{-1}(x)\, R^{-1}(x)\, T(x+1)$$

$$= I + \frac{1}{x^{\frac{k-1}{2}}} \left\| \begin{matrix} \psi_{11}(x) & e^{-4\sqrt{\alpha}\sqrt{x}}\, \psi_{12}(x) \\ e^{4\sqrt{\alpha}\sqrt{x}}\, \psi_{21}(x) & \psi_{22}(x) \end{matrix} \right\|,$$

and we need to choose $k \geqq 5$. If we write $e^{2\sqrt{\alpha}} = \varrho_1$, $e^{-2\sqrt{\alpha}} = \varrho_2$, the formulas become still more like those of the general case; by our choice of $\sqrt{\alpha}$ above we have $R(\sqrt{\alpha}) > 0$, so $|\varrho_1| > |\varrho_2|$.

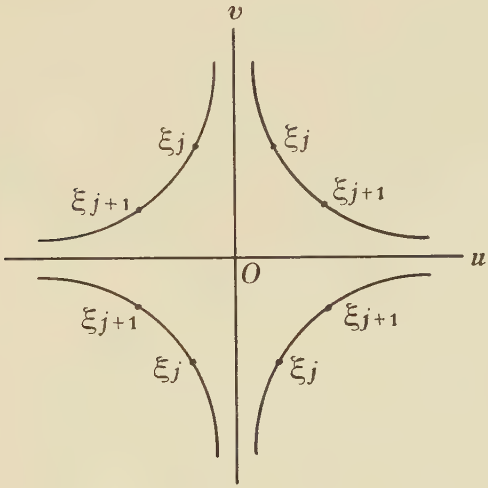


Fig. 21.

The formula which corresponds to (117) requires investigation; the product to be considered is

$$\left| \left(\frac{\varrho_\gamma}{\varrho_i}\right)^{\sqrt{x+r}} \left(\frac{\varrho_\delta}{\varrho_\gamma}\right)^{\sqrt{x+s}} \cdots \left(\frac{\varrho_2}{\varrho_\eta}\right)^{\sqrt{x+w}} \right|,$$

which may be written in the form

$$(224) \qquad \left| \left(\frac{\varrho_2}{\varrho_i}\right)^{\xi_1} \left(\frac{\varrho_2}{\varrho_\gamma}\right)^{\xi_2-\xi_1} \left(\frac{\varrho_2}{\varrho_\delta}\right)^{\xi_3-\xi_2} \cdots \left(\frac{\varrho_2}{\varrho_\eta}\right)^{\xi_l-\xi_{l-1}} \right|,$$

where $\xi_1 = \sqrt{x+r}$, $\xi_2 = \sqrt{x+s}$, $\cdots$, $\xi_l = \sqrt{x+w}$. Some of the subscripts $\gamma, \delta, \cdots, \eta$ are 2, and the corresponding factors are equal to 1 in absolute value, so we need to consider only the factors

$$\left(\frac{\varrho_2}{\varrho_1}\right)^{\xi_{j+1}-\xi_j} = e^{-4\sqrt{\alpha}\,(\xi_{j+1}-\xi_j)};$$

these will be less than 1 in absolute value if the real part of the exponent of $e$ is negative.

It is necessary to consider carefully what determination of the radicals is to be used. Since $r, s, \cdots, w$ are an increasing set of positive integers, the points $x, x+r, x+s, \cdots, x+w$ lie on a straight line parallel to the axis of reals, and the corresponding points $\xi_0, \xi_1, \xi_2, \cdots, \xi_l$ lie on a rectangular hyperbola in the first and third or second and fourth quadrants, according as $x$ is in the upper or lower half plane. As the subscripts increase, the points $\xi_j$ approach nearer and nearer to the axis of reals. Let us take $-\pi < \arg x < \pi$ and $\arg \sqrt{x} = \frac{1}{2} \arg x$; then $\xi_j$ is in the first or fourth quadrant, and the real part of $\xi_{j+1} - \xi_j$ is positive, while its imaginary part is negative or positive according as $x$ is in the upper or lower half plane (cf. Fig. 21). Write $\xi_{j+1} - \xi_j = u + iv$; since $0 \leqq \arg \sqrt{\alpha} < \pi/2$, we may write

$$4\sqrt{\alpha} = a + ib \qquad (a > 0,\ b \geqq 0);$$

then we have for the exponent of $e$:

$$-4\sqrt{\alpha}\,(\xi_{j+1} - \xi_j) = -au + bv - i(bu + av).$$

If $x$ is in the upper half plane, $v < 0$ and the real part of the exponent is negative, so all the factors in (224) after the first are less than 1 in absolute value. If $x$ is in the lower half plane, $v > 0$, and the real part of the exponent is positive or negative according as $v/u$ is greater or less than $a/b$. If we draw a straight line from the origin in the fourth quadrant making the angle $\tan^{-1}(a/b)$ with the positive axis of reals, it will cut the hyperbola at the point where $v/u = a/b$; when $\sqrt{x}$ lies to the right of this line all the factors in (224) after the first will be less than 1 in absolute value. Hence when $x$ is in the upper half plane or in the lower half plane to the right of the straight line joining the origin to the point $-\alpha$ (which makes the angle $\pi - \arg \alpha = 2\tan^{-1}(a/b)$ with the positive axis of reals), we have

$$\left| \left(\frac{\varrho_\gamma}{\varrho_i}\right)^{\xi_1} \left(\frac{\varrho_\delta}{\varrho_\gamma}\right)^{\xi_2} \cdots \left(\frac{\varrho_2}{\varrho_\eta}\right)^{\xi_l} \right| \leqq \left| \left(\frac{\varrho_2}{\varrho_i}\right)^{\xi_1} \right| \leqq \left| \left(\frac{\varrho_2}{\varrho_i}\right)^{\sqrt{x}} \right|.$$

The second inequality here follows from the fact that

$$\left(\frac{\varrho_2}{\varrho_1}\right)^{\sqrt{x+r}} = \left(\frac{\varrho_2}{\varrho_1}\right)^{\sqrt{x}} e^{-4\sqrt{\alpha}\,(\sqrt{x+r}-\sqrt{x})} = \left(\frac{\varrho_2}{\varrho_1}\right)^{\sqrt{x}} e^{-4\sqrt{\alpha}\,(\xi_1-\xi_0)},$$

where the real part of the exponent of $e$ is negative or zero.

The rest of the argument is essentially the same as in the general case; the part of the lower half plane to the left of the line from 0 through $-\alpha$ can be treated like the portion of the plane not in the region $D$ of fig. 1. The elements of the second column of $H_n(x)$ converge uniformly as $n \to \infty$ to limit functions $h_{12}(x)$, $h_{22}(x)$ which are analytic everywhere in the plane except for poles at $x = 0, -1, -2, \cdots$, which form a solution of the system (222), and which are represented asymptotically by $s_{12}(x)$, $s_{22}(x)$ in the sector $\arg \alpha - \pi < \arg x < \pi$. The determinant $|H_n(x)|$ converges uniformly to the limit function

$$D(x) = -2\sqrt{\alpha}\, s_1\, s_2\, \varrho^{2x+1} \frac{\Gamma(x)}{\Gamma(x+\beta+1)},$$

which is analytic except for poles at $x = 0, -1, -2, \cdots$, and is represented asymptotically by $|S(x)|$ in the sector $-\pi < \arg x < \pi$.

Similarly, starting with the product $G_n(x)$ [eq. (114)], and taking $0 \leq \arg \sqrt{x} < \pi$, we find that the elements of the first column converge uniformly to limit functions $g_{11}(x)$, $g_{21}(x)$ which are analytic except for poles at $x = -\beta, 1-\beta, 2-\beta, \cdots$, which form a solution of (222), and which are represented asymptotically by $s_{11}(x)$, $s_{21}(x)$ in the sector $0 < \arg x < \arg \alpha + \pi$. The determinant $|G_n(x)|$ converges uniformly to the limit function

$$\bar{D}(x) = -2\sqrt{\alpha}\, s_1\, s_2\, \varrho^{2x+1} \frac{\bar{\Gamma}(x)}{\bar{\Gamma}(x+\beta+1)},$$

which is analytic except for poles at $x = -\beta, 1-\beta, 2-\beta, \cdots$, and is represented asymptotically by $|S(x)|$ in the sector $0 < \arg x < 2\pi$.

If we use the other determination of $\sqrt{x}$, $\sqrt{x+r}$, etc., so that $\xi_j$ is in the second or third quadrant, then the elements

of the *first* column of $H_n(x)$ converge to limit functions which are represented asymptotically by $s_{11}(x)$, $s_{21}(x)$ in the sector $\arg\alpha - \pi < \arg x < \pi$. Since, however, the series $s_{11}(x)$, $s_{21}(x)$ are interchanged with $s_{12}(x)$, $s_{22}(x)$ when $\sqrt{x}$ is replaced by $-\sqrt{x}$, the solution obtained in this way has the same properties as $h_{12}(x)$, $h_{22}(x)$. Similar remarks apply to $G_n(x)$.

No important changes are required if we allow $\alpha$ to be in the lower half plane instead of the upper. The rôles of the upper and lower half planes are merely interchanged; e. g.. $h_{12}(x) \sim s_{12}(x)$ in the sector $-\pi < \arg x < \arg\alpha + \pi$ in this case ($\arg\alpha$ being taken between 0 and $-\pi$).

It appears to be impossible to extend to the case $\varrho_1 = \varrho_2$ with any completeness the proofs of the existence of intermediate and principal solutions such as we had in the previous cases. While some results in this direction can be obtained, they are so fragmentary that we will not take them up, but pass on to the integral and series solutions of eq. (221).*

Applying the Laplace transformation (138), we find that eq. (221) is satisfied by the integral

$$y(x) = \int_a^b t^{x-1}(t-\varrho)^{\beta-1} e^{\frac{\alpha\varrho}{t-\varrho}}\, dt,$$

provided the limits are so chosen that

$$\left[t^x (t-\varrho)^{\beta+1} e^{\frac{\alpha\varrho}{t-\varrho}}\right]_a^b = 0.$$

This expression vanishes at $t=0$ if $R(x) > 0$; at $t=\infty$ if $R(x+\beta+1) < 0$; and at $t=\varrho$, provided $t \to \varrho$ in such a way that $R(\alpha\varrho/(t-\varrho)) \to -\infty$; if we draw a line through $t=\varrho$ perpendicular to the line joining $\varrho$ to $(\alpha+1)\varrho$, $t$ must approach $\varrho$ on the side of this line opposite to $(\alpha+1)\varrho$.

Taking the limits $0, \varrho$, we obtain the solution

$$h(x) = \int_0^\varrho t^{x-1}(t-\varrho)^{\beta-1} e^{\frac{\alpha\varrho}{t-\varrho}}\, dt,$$

* Cf. a forthcoming paper by C. R. Adams: On the Irregular Cases of the Linear Ordinary Difference Equation. in *Trans. Amer. Math. Soc.*, probably vol. 30 (1928).

which is valid if $R(x) > 0$. If $\arg \alpha < \pi/2$, we can take the path of integration as the straight line from 0 to $\varrho$ (Fig. 22); otherwise we will let it follow the straight line to a point near $\varrho$, and then make a partial positive circuit about $\varrho$,

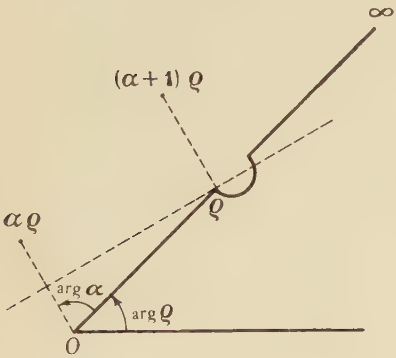

Fig. 22.

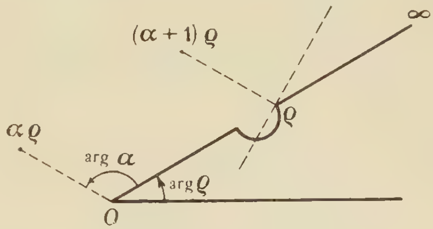

Fig. 23.

ending there in the half plane opposite $(\alpha+1)\varrho$ (Fig. 23). On the rectilinear part of the path we will take $\arg t = \arg \varrho$ and $\arg(t-\varrho) = \arg \varrho + \pi$.

If we take for the path of integration a loop $\lambda$ starting and ending at $t = \varrho$ in the half plane opposite $(\alpha+1)\varrho$ and passing around $t = 0$ in the positive direction, we have if $R(x) > 0$

$$(225) \qquad \int_\lambda t^{x-1}(t-\varrho)^{\beta-1} e^{\frac{\alpha\varrho}{t-\varrho}}\, dt = (e^{2\pi i x} - 1)\, h(x).$$

Since this integral is analytic for all values of $x$, we may use (225) as the definition of $h(x)$ when $R(x) \leq 0$. We see that $h(x)$ has simple poles at $x = 0, -1, -2, \cdots$.

If we set $t = \varrho\tau$, $h(x)$ takes the form

$$(226) \quad h(x) = (-\varrho)^{\beta-1} \varrho^x \int_0^1 \tau^{x-1}(1-\tau)^{\beta-1} e^{-\frac{\alpha}{1-\tau}}\, d\tau.$$

Taking the limits $\infty$, $\varrho$, we have the solution

$$g(x) = \int_\infty^\varrho t^{x-1}(t-\varrho)^{\beta-1} e^{\frac{\alpha\varrho}{t-\varrho}}\, dt,$$

which is valid if $R(x+\beta)<1$. If $\arg\alpha>\pi/2$, we can take for the path of integration the prolongation of the ray from 0 to $\varrho$ (Fig. 23); otherwise we will let it follow this line to a point near $\varrho$, then make a partial negative circuit about $\varrho$, ending there in the half plane opposite $(\alpha+1)\varrho$ (Fig. 22). On the straight line we will take $\arg t = \arg(t-\varrho) = \arg\varrho$.

If we take for the path of integration a loop $A$ which starts and ends at $t=\varrho$ in the half plane opposite $(\alpha+1)\varrho$ and passes around $t=\infty$ in the positive direction (i. e., around $t=0$ in the negative direction), we have if $R(x+\beta)<1$

$$(227)\quad \int_A t^{x-1}(t-\varrho)^{\beta-1}e^{\frac{\alpha\varrho}{t-\varrho}}\,dt = (e^{-2\pi i(x+\beta)}-1)\,g(x).$$

Since this integral is analytic for all values of $x$, we may use (227) as the definition of $g(x)$ when $R(x+\beta)\geqq 1$. We see that $g(x)$ has simple poles at $x=1-\beta,\ 2-\beta,\ 3-\beta, \cdots$.

If we set $t=\varrho/\tau$, and introduce the factor $e^{\alpha}$ into the integrand, we have

$$(228)\quad g(x) = -e^{-\alpha}\varrho^{x+\beta-1}\int_0^1 \tau^{-x-\beta}(1-\tau)^{\beta-1}e^{\frac{\alpha}{1-\tau}}\,d\tau.$$

If we take for the path of integration a curve $K$ which starts at $t=\varrho$ in the half plane opposite $(\alpha+1)\varrho$, makes a small positive circuit about $\varrho$, and returns in the same half plane, we obtain the solution

$$l(x) = \int_K t^{x-1}(t-\varrho)^{\beta-1}e^{\frac{\alpha\varrho}{t-\varrho}}\,dt.$$

We will take $\arg t$ near $\arg\varrho$, and $\arg(t-\varrho)$ between $\arg(\alpha\varrho)-\pi$ and $\arg(\alpha\varrho)+\pi$.

If we set $t=\varrho(1-\alpha\tau)$, or $t-\varrho=-\alpha\varrho\tau=e^{\pi i}\alpha\varrho\tau$,

$$l(x) = (-\alpha)^{\beta}\varrho^{x+\beta-1}\int_{K'}\tau^{\beta-1}(1-\alpha\tau)^{x-1}e^{-\frac{1}{\tau}}\,d\tau,$$

where $K'$ is a curve similar to $K$ about $\tau=0$ (cf. Fig. 4), and $\arg\tau$ has values between $-2\pi$ and 0. Let $K$ be small

enough so that $|\alpha\tau| < 1$ at all points on $K'$; then we can expand the second factor of the integrand by the binomial theorem and integrate term by term by means of (72):

$$l(x) = e^{\pi i\beta}\alpha^\beta \varrho^{x+\beta-1}\Big[-\overline{\Gamma}(-\beta) + (x-1)\,\alpha\overline{\Gamma}(-\beta-1) - \frac{(x-1)(x-2)}{1\cdot 2}\alpha^2\overline{\Gamma}(-\beta-2) + \cdots\Big]$$

$$= -e^{\pi i\beta}\alpha^\beta \varrho^{x+\beta-1}\,\overline{\Gamma}(-\beta)\Big[1 + \frac{(x-1)\,\alpha}{1\cdot(\beta+1)} + \frac{(x-1)(x-2)\,\alpha^2}{1\cdot 2\,(\beta+1)(\beta+2)} + \cdots\Big]$$

$$= 2\pi i\,\alpha^\beta\frac{\varrho^{x+\beta-1}}{\Gamma(\beta+1)}\Big[1 + \frac{(x-1)\,\alpha}{1\cdot(\beta+1)} + \frac{(x-1)(x-2)\,\alpha^2}{1\cdot 2\,(\beta+1)(\beta+2)} + \cdots\Big].$$

This solution may also be written in the form

$$l(x) = 2\pi i\,\alpha^\beta e^{-\alpha}\frac{\varrho^{x+\beta-1}}{\Gamma(\beta+1)}\Big[1 + \frac{(x+\beta)\,\alpha}{1\cdot(\beta+1)} + \frac{(x+\beta)(x+\beta+1)\,\alpha^2}{1\cdot 2\,(\beta+1)(\beta+2)} + \cdots\Big].$$

These series converge for all values of $x$, $\alpha$, and $\beta$ except when $\beta$ is a negative integer, in which case the solution is, however, analytic. We see that $l(x)$ is an entire function.

Taking the limits $0$, $\infty$, we obtain the solution

$$m(x) = \int_0^\infty t^{x-1}(t-\varrho)^{\beta-1}e^{\frac{\alpha\varrho}{t-\varrho}}\,dt,$$

which is valid if $R(x) > 0$ and $R(x+\beta) < 1$. The path of integration we can take as any ray from $0$ to $\infty$, except the one through $t = \varrho$; in particular, we can take the ray through $-\varrho$. Let $\arg t$ have a value between $\arg\varrho$ and $\arg\varrho - 2\pi$, and let $\arg(t-\varrho)$ be between $\arg\varrho$ and $\arg\varrho + 2\pi$.

If we set $t = -\varrho\tau/(1-\tau)$
we have

$$m(x) = e^{-\alpha}(e^{-\pi i}\varrho)^x (e^{\pi i}\varrho)^{\beta-1}\int_0^1 \tau^{x-1}(1-\tau)^{-x-\beta} e^{\alpha\tau}\, d\tau$$

$$= -e^{-\alpha} e^{\pi i(\beta-x)} \varrho^{x+\beta-1}\int_0^1 \tau^{x-1}(1-\tau)^{-x-\beta} \times\left(1+\alpha\tau+\frac{\alpha^2\tau^2}{2!}+\cdots\right)d\tau$$

$$= -e^{-\alpha} e^{\pi i(\beta-x)} \varrho^{x+\beta-1}[B(x,\, 1-x-\beta) + \alpha B(x+1,\, 1-x-\beta)+\cdots]$$

$$(229)\quad \begin{cases} = -e^{-\alpha} e^{\pi i(\beta-x)} \varrho^{x+\beta-1} B(x,\, 1-x-\beta)\left[1+\dfrac{x\alpha}{1\cdot(1-\beta)}\right. \\ \qquad\left. + \dfrac{x(x+1)\alpha^2}{1\cdot 2(1-\beta)(2-\beta)}+\cdots\right]. \end{cases}$$

If we leave the factor $e^{-\alpha}$ in the integrand, we get the solution in the form

$$(230)\quad \begin{aligned} m(x) &= -e^{\pi i(\beta-x)} \varrho^{x+\beta-1} B(x,\, 1-x-\beta) \\ &\times\left[1+\frac{(x+\beta-1)\alpha}{1\cdot(1-\beta)}+\frac{(x+\beta-1)(x+\beta-2)\alpha^2}{1\cdot 2(1-\beta)(2-\beta)}+\cdots\right]. \end{aligned}$$

The series in brackets in (229) and (230) converge for all values of $x$, $\alpha$, and $\beta$ except when $\beta$ is a positive integer, in which case the solution is, however, analytic. We may use (229) or (230) as the definition of $m(x)$ when the integral is not valid. We see that $m(x)$ has poles at $x = 0, -1, -2, \cdots$ and at $x = 1-\beta,\ 2-\beta,\ 3-\beta, \cdots$.

We can express $h(x)$ and $g(x)$ by means of partial fraction series, as in the general case. Let $b$ be any point on the straight line segment joining 0 to $\varrho$, and integrate over the contour consisting of the segment $0b$, a loop $k$ about $\varrho$, traversed in the positive direction, and the segment $b0$, starting with $\arg t = \arg\varrho$ and $\arg(t-\varrho) = \arg\varrho + \pi$. If $R(x) > 0$, the integral over this contour is equal to $(1-e^{2\pi i\beta})\, h(x) + e^{2\pi i\beta}\, l(x)$; hence

$$(1 - e^{2\pi i\beta})\, h(x) + e^{2\pi i\beta}\, l(x)$$
$$= (1 - e^{2\pi i\beta}) \int_0^b t^{x-1}\, v(t)\, dt + \int_k t^{x-1}\, v(t)\, dt.$$

The last integral is an entire function, which we may combine with $-e^{2\pi i\beta}\, l(x)$ and write $(1 - e^{2\pi i\beta})\, E(x)$. Hence if $\beta$ is not an integer

$$h(x) = E(x) + \int_0^b t^{x-1}\, (t - \varrho)^{\beta-1}\, e^{\frac{\alpha\varrho}{t-\varrho}}\, dt.$$

Introducing the factor $e^{\alpha}$ into the integrand and expanding in powers of $t$, we have

$$h(x) = E(x) + e^{-\alpha}\, (-\varrho)^{\beta-1} \int_0^b t^{x-1} \Big[1 - \frac{\alpha+\beta-1}{\varrho}\, t$$
$$+ \frac{(\alpha+\beta)^2 - 4\alpha - 3\beta + 2}{2\varrho^2}\, t^2 - \cdots\Big]\, dt$$
$$= E(x) + e^{-\alpha}\, (-\varrho)^{\beta-1}\, b^x \Big[\frac{1}{x} - \frac{\alpha+\beta-1}{\varrho}\, \frac{b}{x+1}$$
$$+ \frac{(\alpha+\beta)^2 - 4\alpha - 3\beta + 2}{2\varrho^2}\, \frac{b^2}{x+2} - \cdots\Big].$$

As in the general case, this series converges uniformly in the neighborhood of every point except $x = 0, -1, -2, \cdots$, and hence represents $h(x)$ even when $R(x) \leqq 0$.

Similarly, if we let $c$ be any point on the straight line $\varrho\infty$ and integrate over the contour consisting of the line $\infty c$, a loop $k'$ about $\varrho$, traversed in the positive direction, and the line $c\infty$, starting with $\arg t = \arg(t - \varrho) = \arg \varrho$, we find that if $\beta$ is not an integer

$$g(x) = E'(x) + c^{x+\beta-1} \Big\{\frac{1}{x+\beta-1} + \frac{(\alpha-\beta+1)\varrho}{c(x+\beta-2)}$$
$$+ \frac{[(\alpha-\beta)^2 + 4\alpha - 3\beta + 2]\,\varrho^2}{2c^2(x+\beta-3)} + \cdots\Big\},$$

where $E'(x)$ is an entire function. This series converges uniformly in the neighborhood of every point except $x = 1-\beta$, $2-\beta$, $3-\beta$, $\cdots$.

To obtain the relations between the solutions $h(x)$, $g(x)$, $l(x)$, and $m(x)$, let us integrate over the contour $ABCDEFA$ of Fig. 24, starting with $\arg t = \arg \varrho$ and $\arg(t-\varrho) = \arg \varrho + \pi$; letting the radii of the arcs $BC$ and $AF$ approach zero and that of $DE$ increase indefinitely, we have in the limit

$$h(x) - e^{2\pi i\beta}\, g(x) - m(x) = 0. \tag{231}$$

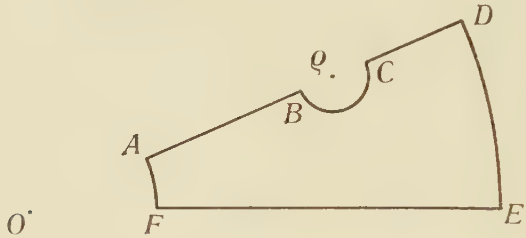


Fig. 24.

Let us now integrate over the contour $ABCDEFGHIJA$ of Fig. 25, taking $\arg t = \arg(t-\varrho) = \arg\varrho - \pi$ on $AB$;

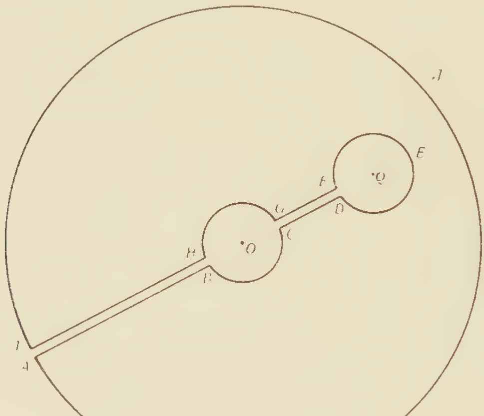

Fig. 25

letting the radii of the arcs $BC$ and $GH$ approach zero and that of $IJA$ increase indefinitely, and letting the points $D$

and $F$ approach $\varrho$ in the half plane opposite $(\alpha+1)\varrho$, we have in the limit

$$-e^{-2\pi i\beta}\, m(x) + e^{-2\pi i\beta}\, h(x) + l(x) - h(x) + e^{2\pi i x}\, m(x) = 0,$$

or, if we multiply by $e^{2\pi i\beta}$ and combine terms,

$$(232)\quad (1-e^{2\pi i\beta})\, h(x) = -e^{2\pi i\beta}\, l(x) + (1-e^{2\pi i(x+\beta)})\, m(x).$$

Eliminating $h(x)$ from (231) and (232), we have

$$(233)\qquad (1-e^{2\pi i\beta})\, g(x) = -l(x) + (1-e^{2\pi i x})\, m(x).$$

Equations (232) and (233) give us the expressions for $h(x)$ and $g(x)$ in terms of $l(x)$ and $m(x)$, provided $\beta$ is not an integer. If $\beta$ is an integer, the solutions $l(x)$ and $m(x)$ are linearly dependent; we have namely in this case

$$l(x) = (1-e^{2\pi i x})\, m(x).$$

Analogy with the previous cases suggests that the solutions $h(x)$ and $g(x)$ are equal to $h_{12}(x)$ and $g_{11}(x)$ respectively. To prove that this is true we will use a method similar to that employed in § 2, Chap. II to identify $\Gamma(x)$ with its integral expression.

From the asymptotic forms of $g_{11}(x)$ and $h_{12}(x)$ we have for large values of $x$ in the upper half plane

$$(234)\qquad \begin{cases} g_{11}(x) = \varrho^x\, e^{2\sqrt{\alpha}\sqrt{x}}\, x^{-\frac{1}{4}-\frac{\beta}{2}}\, [s_1+\varepsilon_1(x)], \\ h_{12}(x) = \varrho^x\, e^{-2\sqrt{\alpha}\sqrt{x}}\, x^{-\frac{1}{4}-\frac{\beta}{2}}\, [s_2+\varepsilon_2(x)], \end{cases}$$

where $\lim\limits_{x\to\infty} \varepsilon_j(x) = 0$ $(j = 1, 2)$. We may write

$$h(x) = p(x)\, g_{11}(x) + q(x)\, h_{12}(x),$$

where $p(x)$ and $q(x)$ are periodic functions. Let us take $\arg\alpha < \pi/2$; then the path of integration in (226) is the axis of reals from 0 to 1. Let $x\to\infty$ along a line parallel to the positive axis of reals and far above it; the integrand

in (226) approaches zero; on account of the factor $\tau^{x-1}$; but $e^{2\sqrt{\alpha}\sqrt{x}}$ increases exponentially; hence $p(x) \rightarrow 0$, i. e., $p(x) = 0$.

Let us investigate the absolute value of the integral in (226), taking $R(x) > 1$. Since the length of the interval is 1, the absolute value of the integral is less than the maximum absolute value of the integrand. Since the integrand vanishes at both ends of the interval, it has its maximum value at some interior point. Let $u$, $a$, $b$ denote the real parts of $x$, $\alpha$, $\beta$ respectively; then the absolute value of the integrand is

$$\tau^{u-1}(1-\tau)^{b-1} e^{-\frac{a}{1-\tau}}.$$

Equating the derivative of this to zero, we obtain the algebraic equation

$$(u+b-2)\tau^2 - (2u+a+b-3)\tau + u - 1 = 0,$$

and if $u > 2 - b$ the root between 0 and 1 is

$$\tau = \frac{2u+a+b-3}{2(u+b-2)}\left[1 - \sqrt{1 - \frac{4(u-1)(u+b-2)}{(2u+a+b-3)^2}}\right]$$

$$\doteq 1 - \frac{\sqrt{a}}{\sqrt{u}} + \frac{a-b+1}{2u} + \cdots.$$

If we use just the first two terms of this, and replace $a$ and $u$ by $\alpha$ and $x$, we have an approximation which is close for large values of $x$. Hence the absolute value of the integrand (and so of the integral) is less than

$$K\left|\left(1 - \frac{\sqrt{\alpha}}{\sqrt{x}}\right)^{x-1}\left(\frac{\sqrt{\alpha}}{\sqrt{x}}\right)^{\beta-1} e^{-\sqrt{\alpha}\sqrt{x}}\right|,$$

where $K$ is a suitably chosen constant. We may write

$$\left(1 - \frac{\sqrt{\alpha}}{\sqrt{x}}\right)^{x-1} = e^{(x-1)\log\left(1 - \frac{\sqrt{\alpha}}{\sqrt{x}}\right)}$$

$$= e^{(x-1)\left(-\frac{\sqrt{\alpha}}{\sqrt{x}} - \frac{\alpha}{2x} - \frac{\alpha^{3/2}}{3x^{3/2}} - \cdots\right)}$$

$$= e^{-\sqrt{\alpha}\sqrt{x}}\, e^{-\frac{\alpha}{2}}\left(1 + \frac{3\sqrt{\alpha} - \alpha^{\frac{3}{2}}}{3\sqrt{x}} + \cdots\right);$$

hence for sufficiently large values of $x$ in the right half plane

$$\left| \int_0^1 \tau^{x-1} (1-\tau)^{\beta-1} e^{-\frac{\alpha}{1-\tau}} d\tau \right| < K' \left| e^{-2\sqrt{\alpha}\sqrt{x}} x^{\frac{1}{2}-\frac{\beta}{2}} \right|,$$

where $K'$ is a constant. Moreover, since we are taking $\alpha$ in the first quadrant, the second equation in (234) holds throughout the fourth quadrant, so we have

$$|\varrho^{-x} h_{12}(x)| > \frac{1}{2} |s_2| \cdot \left| e^{-2\sqrt{\alpha}\sqrt{x}} x^{-\frac{1}{4}-\frac{\beta}{2}} \right|$$

for sufficiently large values of $x$ in the right half plane.

Consider now the periodic function $q(x) = h(x)/h_{12}(x)$; both $h(x)$ and $h_{12}(x)$ are single-valued and analytic in the right half plane, and $h_{12}(x)$ does not vanish for large values of $x$ there, as we see from (234); hence if we take a period strip far enough to the right, $q(x)$ is single-valued and analytic throughout the finite part of it. The inequalities above show that at the ends of the strip

$$|q(x)| < K'' \left| x^{\frac{3}{4}} \right|,$$

where $K''$ is a constant. Writing $q(x) = Q(z)$, where $z = e^{2\pi i x}$, we see that

$$|z Q(z)| < K'' |z| \cdot \left| \frac{\log z}{2\pi i} \right|^{\frac{3}{4}};$$

hence $z Q(z)$ is analytic at $z = 0$ and vanishes there; similarly, $Q(z)/z$ is analytic and vanishes at $z = \infty$. Accordingly $Q(z)$ has no pole in the entire $z$-plane, so it must be a constant.

This proves that $h(x)$ differs from $h_{12}(x)$ only by a constant factor, and by giving the proper value to $s_2$ we can write $h(x) = h_{12}(x)$. Both sides of this equation are analytic functions of $\alpha$ as well as of $x$, so the result holds when $\alpha$ is not restricted to the first quadrant. A similar investigation of the integral in (228) shows that if the proper value is given to $s_1$, $g(x) = g_{11}(x)$.

This identification of $h(x)$ and $g(x)$ with $h_{12}(x)$ and $g_{11}(x)$ shows that $h(x) \sim S_2(x)$ in the sector $\arg \alpha - \pi < \arg x < \pi$

and $g(x) \sim S_1(x)$ in the sector $0 < \arg x < \arg \alpha + \pi$.* Since these sectors both include the upper half plane, we can obtain the asymptotic forms of $l(x)$ and $m(x)$ there.

By eqs. (231) and (233),

$$l(x) = (1 - e^{2\pi i x})\, h(x) - (1 - e^{2\pi i(x+\beta)})\, g(x);$$

hence in the upper half plane

$$l(x) \sim S_2(x) - S_1(x).$$

The dominant term depends on the factors $e^{-2\sqrt{\alpha}\sqrt{x}}$, $e^{2\sqrt{\alpha}\sqrt{x}}$, or on the exponents $-4\sqrt{\alpha}\sqrt{x}, 0$. If we write $4\sqrt{\alpha} = a + ib$ $(a > 0,\ b \geq 0)$ and $\sqrt{x} = u + iv$ $(u > 0,\ v > 0)$, the real part of the first exponent is $-au + bv$; this is positive if $v/u > a/b$, or $\arg x > \pi - \arg \alpha$, and negative if $v/u < a/b$, or $\arg x < \pi - \arg \alpha$; hence

$$l(x) \sim \begin{cases} -S_1(x), & 0 < \arg x < \pi - \arg \alpha, \\ S_2(x), & \pi - \arg \alpha < \arg x < \pi. \end{cases}$$

By eq. (231),

$$m(x) \sim S_2(x) - e^{2\pi i \beta} S_1(x)$$

in the upper half plane. The exponents are the same as for $l(x)$, so

$$m(x) \sim \begin{cases} -e^{2\pi i \beta} S_1(x), & 0 < \arg x < \pi - \arg \alpha, \\ S_2(x), & \pi - \arg \alpha < \arg x < \pi. \end{cases}$$

If $\alpha \neq 0$, eq. (221) is never reducible, for if it had a solution in common with an equation of the form (38), this would be represented asymptotically by a series of the form (44), which would be formally a solution of eq. (221); but eq. (221) is not satisfied by any series of this form.

If $\alpha = 0$, as noted at the beginning of this section, the series $S_1(x)$ and $S_2(x)$ break down, but eq. (221) is satisfied by two series of the same form as in the general case, namely

* For another method of obtaining the asymptotic forms of the integral solutions of eq. (221), which is too long and complicated to be given here, but which leads to more extensive results, see a paper by Galbrun, *Bull. Soc. Math. de France*, 49 (1921), pp. 206–241,

$$S'(x) = \varrho^x s_1',$$
$$S''(x) = \varrho^x x^{-\beta} s_2' \left[ 1 - \frac{\beta(\beta-1)}{2x} + \cdots \right],$$

where $s_1'$ and $s_2'$ are arbitrary constants; all the coefficients in $S'(x)$ vanish except the first.

One analytic solution is obviously $s_1' \varrho^x$; two others are obtained if we set $\alpha = 0$ in eqs. (226) and (228), namely

$$h'(x) = (-\varrho)^{\beta-1} \varrho^x \int_0^1 \tau^{x-1} (1-\tau)^{\beta-1} d\tau = (-\varrho)^{\beta-1} \varrho^x B(x, \beta),$$

$$g'(x) = \varrho^{x+\beta-1} \int_0^1 \tau^{-x-\beta} (1-\tau)^{\beta-1} d\tau = \varrho^{x+\beta-1} B(1-x-\beta, \beta);$$

these differ only by a periodic factor:

$$g'(x) = \frac{1 - e^{2\pi i x}}{1 - e^{2\pi i (x+\beta)}} h'(x).$$

If we take $s_2' = (-\varrho)^{\beta-1} \Gamma(\beta)$, then $h'(x) \sim S''(x)$ in the sector $-\pi < \arg x < \pi$, and $g'(x) \sim S''(x)$ in the sector $0 < \arg x < 2\pi$.

Since $\varrho^x$, $h'(x)$, and $g'(x)$ all satisfy equations of the form (38), eq. (221) is reducible. Hence a necessary and sufficient condition that eq. (221) be reducible is that $\alpha = 0$.

## § 4. The case $\varrho_1 = \varrho_2 = 0$.

If in eq. (99) $a_1 = a_0 = 0$, but $a_2 \neq 0$, both roots of the characteristic equation (101) are zero. This case presents even greater difficulties than the previous one, and the results obtained in the present section are far from complete.

If we set

$$\frac{b_2}{a_2} = \gamma + 2, \quad \frac{b_1}{a_2} = -\alpha, \quad \frac{b_0}{a_2} = -\sigma,$$

eq. (99) takes the form

$$(x+\gamma+2)\, y(x+2) - \alpha y(x+1) - \sigma y(x) = 0;$$

if we write $x + \gamma = x'$, $y(x' - \gamma) = f(x')$, then $f(x)$ satisfies the equation

(235) $$(x+2)\,y(x+2) - \alpha y(x+1) - \sigma y(x) = 0,$$

which we will take as our normal form. We may assume that $\alpha$ and $\sigma$ are both different from zero, since otherwise the equation is equivalent to one of the first order.

We find by trial that if we substitute for $y(x)$ a series of the form

$$x^{\mu x}\, a^x\, e^{-\mu x}\, e^{b\sqrt{x}}\, x^c \left(s + \frac{s'}{\sqrt{x}} + \frac{s''}{x} + \cdots\right),$$

the constants can be so chosen that eq. (235) is formally satisfied; we obtain in this way the two series

$$S_1(x) = x^{-\frac{1}{2}x} (\sqrt{\sigma})^x\, e^{\frac{1}{2}x}\, e^{\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}}\, x^{-\frac{1}{2}} \left(s_1 + \frac{s_1'}{\sqrt{x}} + \frac{s_1''}{x} + \cdots\right),$$

$$S_2(x) = x^{-\frac{1}{2}x} (-\sqrt{\sigma})^x\, e^{\frac{1}{2}x}\, e^{-\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}}\, x^{-\frac{1}{2}} \left(s_2 + \frac{s_2'}{\sqrt{x}} + \frac{s_2''}{x} + \cdots\right),$$

where

$$\frac{s_1'}{s_1} = \frac{\alpha^3 + 6\,\alpha\sigma}{24\,\sigma\sqrt{\sigma}}, \quad \frac{s_2'}{s_2} = -\frac{s_1'}{s_1}, \text{ etc.}$$

The series are uniquely determined except for the constant factors $s_1$ and $s_2$. As in § 3,

$$\frac{s_2^{(k)}}{s_2} = (-1)^k \frac{s_1^{(k)}}{s_1},$$

and if we write the series in the form

$$S'(x) = \left(\frac{\sqrt{\sigma}}{\sqrt{x}}\right)^{x+1} e^{\frac{1}{2}x}\, e^{\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}} \left(s + \frac{s'}{\sqrt{x}} + \frac{s''}{x} + \cdots\right),$$

$$S''(x) = \left(-\frac{\sqrt{\sigma}}{\sqrt{x}}\right)^{x+1} e^{\frac{1}{2}x}\, e^{-\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}} \left(s - \frac{s'}{\sqrt{x}} + \frac{s''}{x} - \cdots\right),$$

it is clear that they are interchanged with each other when $x$ makes a circuit about the origin.

The two series are also interchanged if $\sqrt{\sigma}$ is replaced by $-\sqrt{\sigma}$. For definiteness let us take

$$-\pi < \arg\frac{\alpha^2}{\sigma} \leq \pi, \qquad \arg\frac{\alpha}{\sqrt{\sigma}} = \frac{1}{2}\arg\frac{\alpha^2}{\sigma};$$

then

$$-\frac{\pi}{2} < \arg\frac{\alpha}{\sqrt{\sigma}} \leq \frac{\pi}{2}, \qquad \text{or } R\left(\frac{\alpha}{\sqrt{\sigma}}\right) \geq 0.$$

Setting $y(x) = y_1(x)$, $y(x+1) = y_2(x)$, we obtain the system

$$(236) \qquad \begin{cases} y_1(x+1) = y_2(x), \\ y_2(x+1) = \dfrac{\sigma}{x+2}\,y_1(x) + \dfrac{\alpha}{x+2}\,y_2(x), \end{cases}$$

which may be written as a matrix equation

$$(237) \qquad Y(x+1) = R(x)\,Y(x),$$

where

$$R(x) = \begin{Vmatrix} 0 & 1 \\ \dfrac{\sigma}{x+2} & \dfrac{\alpha}{x+2} \end{Vmatrix}.$$

Eq. (237) is satisfied formally by the matrix of series

$$S(x) = x^{-\frac{1}{2}x}\,e^{\frac{1}{2}x}\,x^{-\frac{1}{2}} \times \begin{Vmatrix} (\sqrt{\sigma})^x e^{\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}}(s_{11}+\cdots) & (-\sqrt{\sigma})^x e^{-\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}}(s_{12}+\cdots) \\ (\sqrt{\sigma})^x e^{\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}}(s_{21}+\cdots) & (-\sqrt{\sigma})^x e^{-\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}}(s_{22}+\cdots) \end{Vmatrix},$$

where $s_{11} = s_1$, $s'_{11} = s'_1$, $s_{21} = 0$, $s'_{21} = \sqrt{\sigma}\,s_1$, $s''_{21} = \sqrt{\sigma}\,s'_1 + \alpha s_1/2$, $s_{12} = s_2$, $s'_{12} = s'_2$, $s_{22} = 0$, $s'_{22} = -\sqrt{\sigma}\,s_2$, $s''_{22} = -\sqrt{\sigma}\,s'_2 + \alpha s_2/2$, etc. The determinant of $S(x)$ is

$$|S(x)| = x^{-x}(-\sigma)^x e^x x^{-\frac{3}{2}}\left(d + \frac{d'}{x} + \frac{d''}{x^2} + \cdots\right),$$

where $d = -2\sqrt{\sigma}\, s_1 s_2$, etc., and the inverse matrix is

$$S^{-1}(x) = x^{\frac{1}{2}x} e^{-\frac{1}{2}x} x$$

$$\times \begin{Vmatrix} (\sqrt{\sigma})^{-x} e^{-\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}} (\sigma_{11} + \cdots) & (\sqrt{\sigma})^{-x} e^{-\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}} (\sigma_{12} + \cdots) \\ (-\sqrt{\sigma})^{-x} e^{\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}} (\sigma_{21} + \cdots) & (-\sqrt{\sigma})^{-x} e^{\frac{\alpha\sqrt{x}}{\sqrt{\sigma}}} (\sigma_{22} + \cdots) \end{Vmatrix},$$

where $\sigma_{11} = 0$, $\sigma'_{11} = 1/2s_1$, $\sigma_{21} = 0$, $\sigma'_{21} = 1/2s_2$, $\sigma_{12} = 1/2\sqrt{\sigma}\, s_1$, $\sigma_{22} = -1/2\sqrt{\sigma}\, s_2$, etc.

The work of obtaining analytic solutions of the system (236) from the matrix products $H_n(x)$ and $G_n(x)$ is practically identical with that in § 3; in the formulas $e^{4\sqrt{\alpha}\sqrt{x}}$ is replaced by $(-1)^x e^{\frac{2\alpha\sqrt{x}}{\sqrt{\sigma}}}$, but this does not necessitate any change in the argument. If we write $e^{\frac{\alpha}{\sqrt{\sigma}}} = \varrho_1$, $e^{-\frac{\alpha}{\sqrt{\sigma}}} = \varrho_2$, the product corresponding to (224) has exactly the same form, and can be treated in the same manner. We find that if we take $-\pi/2 < \arg \sqrt{x} < \pi/2$ the elements of the second column of $H_n(x)$ converge uniformly in the neighborhood of every finite point in the plane to analytic functions $h_{12}(x)$, $h_{22}(x)$, which form a solution of (236), and which are represented asymptotically by $s_{12}(x)$, $s_{22}(x)$ in the sector $\arg \alpha^2/\sigma - \pi < \arg x < \pi$ if $\alpha^2/\sigma$ is in the upper half plane, and in the sector $-\pi < \arg x < \arg \alpha^2/\sigma + \pi$ if $\alpha^2/\sigma$ is in the lower half plane; the determinant of $H_n(x)$ converges uniformly to the limit function

$$D(x) = -2\sqrt{2\pi\sigma}\, s_1 s_2 \frac{(-\sigma)^x}{\Gamma(x+2)},$$

which is analytic throughout the finite part of the plane and is represented asymptotically by $|S(x)|$ in the sector $-\pi < \arg x < \pi$. If we use the other determination of $\sqrt{x}$, the elements of the first column of $H_n(x)$ converge to an analytic solution. Similarly, if we take $0 < \arg \sqrt{x} < \pi$,

the elements of the first column of $G_n(x)$ converge uniformly to limit functions $g_{11}(x)$, $g_{21}(x)$ analytic except for poles at $x = -1, 0, 1, 2, \cdots$, which form a solution of (236), and which are represented asymptotically by $s_{11}(x)$, $s_{21}(x)$ in the sector $0 < \arg x < \arg \alpha^2/\sigma + \pi$ if $\alpha^2/\sigma$ is in the upper half plane, and in the sector $\arg \alpha^2/\sigma + \pi < \arg x < 2\pi$ if $\alpha^2/\sigma$ is in the lower half plane. The determinant of $G_n(x)$ converges uniformly to the limit function

$$\overline{D}(x) = -2\sqrt{2\pi\sigma}\, s_1\, s_2 \frac{(-\sigma)^x}{\Gamma(x+2)},$$

which is analytic except for poles at $x = -1, 0, 1, 2, \cdots$, and is represented asymptotically by $|S(x)|$ in the sector $0 < \arg x < 2\pi$.

Applying the Laplace transformation, we find that eq. (235) is satisfied by the integral

$$y(x) = \int_a^b t^{x-1}\, e^{\frac{\alpha}{t}}\, e^{\frac{\sigma}{2t^2}}\, dt,$$

provided the limits are so chosen that

$$\left[ t^{x+2}\, e^{\frac{\alpha}{t} + \frac{\sigma}{2t^2}} \right]_a^b = 0.$$

This expression vanishes at $t = \infty$ if $R(x) < -2$, and at $t = 0$, provided $t \to 0$ in such a way that $R(\alpha/t + \sigma/2t^2) \to -\infty$; this is the case if $t \to 0$ in either of the sectors

$$\frac{1}{2}\arg\sigma - \frac{3\pi}{4} < \arg t < \frac{1}{2}\arg\sigma - \frac{\pi}{4},$$

$$\frac{1}{2}\arg\sigma + \frac{\pi}{4} < \arg t < \frac{1}{2}\arg\sigma + \frac{3\pi}{4},$$

i. e., in the quadrant containing $-i\sqrt{\sigma}$ or $i\sqrt{\sigma}$ in Fig. 26.

Taking for the path of integration a loop $J_1$ which starts at $t = 0$ in the quadrant containing $i\sqrt{\sigma}$, passes through

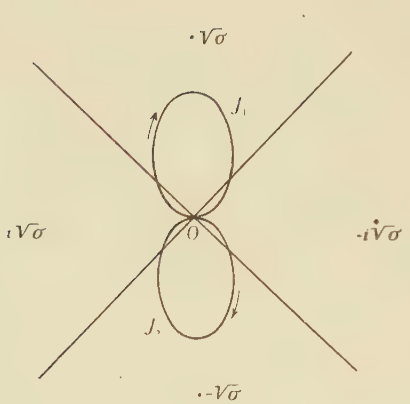

Fig. 26.

the quadrant containing $\sqrt{\sigma}$ in the negative direction, and returns to 0 in the quadrant containing $-i\sqrt{\sigma}$, we obtain the solution

$$h(x) = \int_{J_1} t^{x-1}\, e^{\frac{\alpha}{t}}\, e^{\frac{\sigma}{2t^2}}\, dt.$$

We will let $\arg t$ have values between $\frac{1}{2}\arg\sigma + \pi/2$ and $\frac{1}{2}\arg\sigma - \pi/2$. If we set

$$t = \sqrt{-\frac{\sigma\tau}{2}} = e^{\frac{\pi i}{2}}\sqrt{\frac{\sigma}{2}}\sqrt{\tau},$$

then

$$h(x) = \frac{1}{2}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}} \int_K \tau^{\frac{x}{2}-1} e^{\frac{\varkappa'}{\sqrt{\tau}}}\, e^{-\frac{1}{\tau}}\, d\tau,$$

where $\varkappa' = \alpha\sqrt{2}/\sqrt{-\sigma}$; $K$ is a curve like that in Fig. 4 (p. 53), and $\arg\tau$ has values between 0 and $-2\pi$. Expanding the second factor of the integrand and integrating term by term, we have

$$\begin{aligned} h(x) = \frac{1}{2}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}} \Big[ & \Gamma\left(-\frac{x}{2}\right) + \varkappa'\Gamma\left(\frac{1}{2} - \frac{x}{2}\right) \\ & + \frac{\varkappa'^2}{2!}\Gamma\left(1 - \frac{x}{2}\right) + \cdots \Big] \\ = \frac{1}{2}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}} \Big\{ & \Gamma\left(-\frac{x}{2}\right)\Big[1 - \frac{\varkappa'^2}{2!}\frac{x}{2} + \frac{\varkappa'^4}{4!}\frac{x}{2}\left(\frac{x}{2} - 1\right) \\ & - \frac{\varkappa'^6}{6!}\frac{x}{2}\left(\frac{x}{2} - 1\right)\left(\frac{x}{2} - 2\right) + \cdots\Big] \\ & + \Gamma\left(\frac{1}{2} - \frac{x}{2}\right)\Big[\varkappa' - \frac{\varkappa'^3}{3!}\left(\frac{x}{2} - \frac{1}{2}\right) \\ & + \frac{\varkappa'^5}{5!}\left(\frac{x}{2} - \frac{1}{2}\right)\left(\frac{x}{2} - \frac{3}{2}\right) - \cdots\Big]\Big\}. \end{aligned}$$

If we write $\varkappa = i\varkappa'/\sqrt{2} = \alpha/\sqrt{\sigma}$, $h(x)$ can be put in the form

$$(238)\quad \begin{cases} h(x) = \dfrac{1}{2}\left(-\dfrac{\sigma}{2}\right)^{\frac{x}{2}} \Big\{ \Gamma\left(-\dfrac{x}{2}\right)\Big[1 + \dfrac{\varkappa^2}{2!}x + \dfrac{\varkappa^4}{4!}x(x-2) \\ \qquad\qquad + \dfrac{\varkappa^6}{6!}x(x-2)(x-4) + \cdots\Big] \\ \qquad - i\sqrt{2}\,\Gamma\left(\dfrac{1}{2} - \dfrac{x}{2}\right)\Big[\varkappa + \dfrac{\varkappa^3}{3!}(x-1) \\ \qquad\qquad + \dfrac{\varkappa^5}{5!}(x-1)(x-3) + \cdots\Big]\Big\}. \end{cases}$$

These series converge for all values of $x$, $\alpha$, and $\sigma$; the solution $h(x)$ is an entire function.

Taking similarly a loop $J_2$ which starts at $t = 0$ in the quadrant containing $-i\sqrt{\sigma}$, passes in the negative direction through the quadrant containing $-\sqrt{\sigma}$, and returns to 0 in the quadrant containing $i\sqrt{\sigma}$, we obtain the solution

$$h'(x) = \int_{J_2} t^{x-1}\, e^{\frac{\alpha}{t}}\, e^{\frac{\sigma}{2t^2}}\, dt.$$

We will let $\arg t$ have values between $\frac{1}{2}\arg\sigma + 3\pi/2$ and $\frac{1}{2}\arg\sigma + \pi/2$. Setting

$$t = -\sqrt{-\frac{\sigma\tau}{2}} = e^{\frac{3\pi i}{2}}\sqrt{\frac{\sigma}{2}}\sqrt{\tau},$$

we have

$$h'(x) = \frac{1}{2}e^{\pi i x}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}} \int_K \tau^{\frac{x}{2}-1}\, e^{-\frac{\varkappa'}{\sqrt{\tau}}}\, e^{-\frac{1}{\tau}}\, d\tau.$$

Expanding the second factor of the integrand and integrating, we have

$$h'(x) = \frac{1}{2}e^{\pi i x}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}}\Big[\Gamma\left(-\frac{x}{2}\right) - \varkappa'\Gamma\left(\frac{1}{2} - \frac{x}{2}\right) + \frac{\varkappa'^2}{2!}\Gamma\left(1 - \frac{x}{2}\right) - \cdots\Big]$$

$$
(239)\quad \begin{cases} = \dfrac{1}{2}\, e^{\pi i x}\left(-\dfrac{\sigma}{2}\right)^{\frac{x}{2}} \Big\{ \Gamma\left(-\dfrac{x}{2}\right)\Big[1+\dfrac{\varkappa^2}{2!}\,x \\ \qquad\qquad\qquad\qquad + \dfrac{\varkappa^4}{4!}\,x\,(x-2)+\cdots\Big] \\ + i\sqrt{2}\,\Gamma\left(\dfrac{1}{2}-\dfrac{x}{2}\right)\Big[\varkappa+\dfrac{\varkappa^3}{3!}\,(x-1) \\ \qquad\qquad\qquad\qquad + \dfrac{\varkappa^5}{5!}\,(x-1)\,(x-3)+\cdots\Big]\Big\}. \end{cases}
$$

Instead of the loop $J_1$ or $J_2$ we can use a curve $K_1$ which begins at $t=0$ in the quadrant containing $-i\sqrt{\sigma}$, makes a complete negative circuit about $t=0$, and returns to 0, starting with $\arg t$ near $\frac{1}{2}\arg\sigma+3\pi/2$. $K_1$ is obviously equivalent to $J_2$ followed by $J_1$, so we have for the solution $l(x)$ obtained in this way

$$
(240)\qquad l(x) = h(x)+h'(x).
$$

Similarly, if we use a curve $K_2$ which begins and ends at $t=0$ in the quadrant containing $i\sqrt{\sigma}$, making a negative circuit about $t=0$, and start with $\arg t$ near $\frac{1}{2}\arg\sigma+\pi/2$, we obtain the solution

$$
(241)\qquad l'(x) = h(x)+e^{-2\pi i x}\,h'(x).
$$

Both $l(x)$ and $l'(x)$ are entire functions.

Another solution is obtained if we take for the path of integration a straight line from 0 to $\infty$ in the quadrant containing $i\sqrt{\sigma}$ (in particular, we may take the line through $i\sqrt{\sigma}$); this gives

$$
g(x) = \int_0^{\infty} t^{x-1}\, e^{\frac{\alpha}{t}}\, e^{\frac{\sigma}{2t^2}}\, dt,
$$

in which we will let $\arg t$ have a value between $\frac{1}{2}\arg\sigma+\pi/4$ and $\frac{1}{2}\arg\sigma+3\pi/4$. The integral is valid if $R(x)<0$. If we set

$$
t = \sqrt{-\frac{\sigma}{2u}} = e^{\frac{\pi i}{2}}\sqrt{\frac{\sigma}{2}}\,\frac{1}{\sqrt{u}}.
$$

then

$$g(x) = \frac{1}{2}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}} \int_0^{\infty} \tau^{-\frac{x}{2}-1}\, e^{\varkappa'\sqrt{\tau}}\, e^{-\tau}\, d\tau,$$

where $\arg \tau$ has a value between $\pi/2$ and $-\pi/2$. Expanding the second factor of the integrand and integrating, we have

$$g(x) = \frac{1}{2}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}} \left[\Gamma\left(-\frac{x}{2}\right) + \varkappa'\Gamma\left(\frac{1}{2}-\frac{x}{2}\right) + \frac{\varkappa'^2}{2!}\Gamma\left(1-\frac{x}{2}\right) + \cdots\right]$$

$$(242)\quad \left\{\begin{aligned} &= \frac{1}{2}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}} \Bigg\{\Gamma\left(-\frac{x}{2}\right)\left[1 + \frac{\varkappa^2}{2!}x + \frac{\varkappa^4}{4!}x(x-2) + \cdots\right] \\ &\quad - i\sqrt{2}\,\Gamma\left(\frac{1}{2}-\frac{x}{2}\right)\left[\varkappa + \frac{\varkappa^3}{3!}(x-1) + \frac{\varkappa^5}{5!}(x-1)(x-3) + \cdots\right]\Bigg\}. \end{aligned}\right.$$

Taking similarly a straight line from 0 to $\infty$ in the quadrant containing $-i\sqrt{\sigma}$, we obtain the solution

$$g'(x) = \int_0^{\infty} t^{x-1}\, e^{\frac{\alpha}{t}}\, e^{\frac{\sigma}{2t^2}}\, dt,$$

which is valid for $R(x) < 0$; we will let $\arg t$ have a value between $\frac{1}{2}\arg\sigma - 3\pi/4$ and $\frac{1}{2}\arg\sigma - \pi/4$. If we set

$$t = -\sqrt{-\frac{\sigma}{2\tau}} = e^{\frac{\pi i}{2}}\sqrt{\frac{\sigma}{2}}\,\frac{1}{\sqrt{\tau}},$$

then

$$g'(x) = \frac{1}{2}e^{-\pi i x}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}} \int_0^{\infty} \tau^{-\frac{x}{2}-1}\, e^{-\varkappa'\sqrt{\tau}}\, e^{-\tau}\, d\tau,$$

where $\arg \tau$ has a value between $\pi/2$ and $-\pi/2$. Expanding and integrating, we have

$$g'(x) = \frac{1}{2}e^{-\pi i x}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}}\left[\Gamma\left(-\frac{x}{2}\right) - \varkappa'\Gamma\left(\frac{1}{2}-\frac{x}{2}\right)\right.$$
$$\left. + \frac{\varkappa'^2}{2!}\Gamma\left(1-\frac{x}{2}\right) - \cdots\right]$$

$$(243)\left\{\begin{aligned} &= \frac{1}{2}e^{-\pi i x}\left(-\frac{\sigma}{2}\right)^{\frac{x}{2}}\left\{\Gamma\left(-\frac{x}{2}\right)\left[1+\frac{\varkappa^2}{2!}x\right.\right. \\ &\qquad\left. + \frac{\varkappa^4}{4!}x(x-2)+\cdots\right] \\ &\quad + i\sqrt{2}\,\Gamma\left(\frac{1}{2}-\frac{x}{2}\right)\left[\varkappa+\frac{\varkappa^3}{3!}(x-1)\right. \\ &\qquad\left.\left. + \frac{\varkappa^5}{5!}(x-1)(x-3)+\cdots\right]\right\}. \end{aligned}\right.$$

Eqs. (242) and (243) may be used to define $g(x)$ and $g'(x)$ when $R(x) \geqq 0$; we see that both solutions have simple poles at $x = 0, 1, 2, \cdots$.

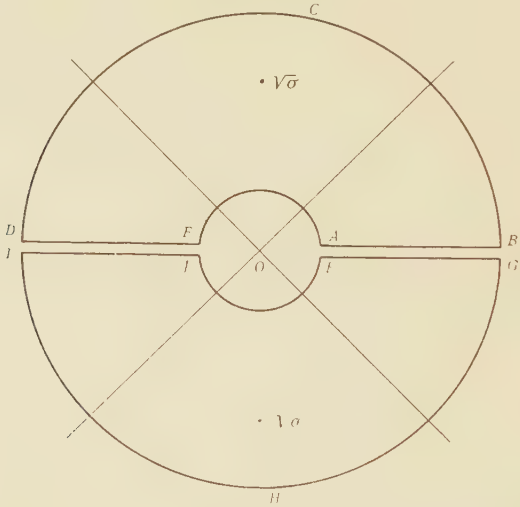


Fig. 27.

We have already expressed $l(x)$ and $l'(x)$ in terms of $h(x)$ and $h'(x)$ [eqs. (240), (241)]. To obtain the relations between

the other integral solutions, let us integrate over the contour $ABCDEA$ of Fig. 27, starting with $\arg t$ near $\frac{1}{2}\arg\sigma - \pi/2$. If $R(x) < 0$ we can let the radius of the arc $BCD$ increase indefinitely; we may also let $A$ and $E$ approach 0 in the sectors containing $-i\sqrt{\sigma}$ and $i\sqrt{\sigma}$ respectively; then we have in the limit

$$g'(x) - g(x) + h(x) = 0,$$

or

(244) $$h(x) = g(x) - g'(x).$$

Similarly, if we integrate over the contour $FGHIJF$, starting with $\arg t$ near $\frac{1}{2}\arg\sigma - \pi/2$, we find that

$$g'(x) - e^{-2\pi ix} g(x) - e^{-2\pi ix} h'(x) = 0,$$

or

(245) $$h'(x) = -g(x) + e^{2\pi ix} g'(x).$$

Since both sides of (244) and (245) are analytic functions of $x$, we can drop the restriction $R(x) < 0$.

Solving these relations for $g(x)$ and $g'(x)$, we have

$$g(x) = -\frac{e^{2\pi ix} h(x) + h'(x)}{1 - e^{2\pi ix}}, \quad g'(x) = -\frac{h(x) + h'(x)}{1 - e^{2\pi ix}}.$$

Another set of integral and series solutions of eq. (235) can be obtained by the same method as in § 1. If we set $y(x) = z(x)/\Gamma(x+1)$ or $z(x)/\overline{\Gamma}(x+1)$, eq. (235) becomes

(246) $$z(x+2) - \alpha z(x+1) - \sigma(x+1) z(x) = 0.$$

The Laplace transformation applied to this yields the solution

$$z(x) = \int_a^b t^x e^{\frac{\alpha t}{\sigma}} e^{-\frac{t^2}{2\sigma}} dt.$$

provided the limits are so chosen that

$$\left[t^{x+1} e^{\frac{\alpha t}{\sigma} - \frac{t^2}{2\sigma}}\right]_a^b = 0.$$

This expression vanishes at $t = 0$ if $R(x) > -1$, and at $t = \infty$, provided $t \to \infty$ in such a way that $R(\alpha t/\sigma - t^2/2\sigma) \to -\infty$; this is the case if $t \to \infty$ in either of the quadrants

$$\frac{1}{2} \arg \sigma - \frac{\pi}{4} < \arg t < \frac{1}{2} \arg \sigma + \frac{\pi}{4},$$

$$\frac{1}{2} \arg \sigma + \frac{3\pi}{4} < \arg t < \frac{1}{2} \arg \sigma + \frac{5\pi}{4},$$

i. e., in the quadrants containing $\sqrt{\sigma}$ or $-\sqrt{\sigma}$.

Taking for the path of integration a line $J_1'$ which starts at $t = \infty$ in the quadrant containing $-\sqrt{\sigma}$, crosses that containing $i\sqrt{\sigma}$, and returns to $\infty$ in that containing $\sqrt{\sigma}$ (Fig. 28), we obtain the solution

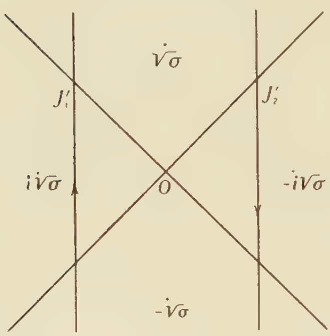

Fig. 28.

$$z_1(x) = \int_{J_1'} t^x e^{\frac{\alpha t}{\sigma}} e^{-\frac{t^2}{2\sigma}} dt.$$

We will let $\arg t$ have values between $\frac{1}{2} \arg \sigma + \pi$ and $\frac{1}{2} \arg \sigma$. If we set $t = \sqrt{2\sigma/\tau}$,

$$z_1(x) = -\frac{1}{2} (2\sigma)^{\frac{x+1}{2}} \int_K \tau^{-\frac{x}{2} - \frac{3}{2}} e^{\sqrt{\frac{2\varkappa}{\tau}}} e^{-\frac{1}{\tau}} d\tau,$$

where $K$ is a curve of the same shape as before, traversed in the negative direction, and $\arg \tau$ has values between $0$ and $-2\pi$. Expanding the second factor of the integrand and integrating, we have

$$z_1(x) = \frac{1}{2} (2\sigma)^{\frac{x+1}{2}} \left[\Gamma\left(\frac{x}{2} + \frac{1}{2}\right) + \sqrt{2\varkappa}\, \Gamma\left(\frac{x}{2} + 1\right) \right.$$
$$\left. + \frac{2\varkappa^2}{2!} \Gamma\left(\frac{x}{2} + \frac{3}{2}\right) + \cdots\right]$$

$$= \frac{1}{2}(2\sigma)^{\frac{x+1}{2}}\left\{\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)\left[1+\frac{\varkappa^2}{2!}(x+1) + \frac{\varkappa^4}{4!}(x+1)(x+3)+\cdots\right] + \sqrt{2}\,\Gamma\left(\frac{x}{2}+1\right)\left[\varkappa+\frac{\varkappa^3}{3!}(x+2) + \frac{\varkappa^5}{5!}(x+2)(x+4)+\cdots\right]\right\}.$$

These series converge for all values of $x$, $\alpha$, and $\sigma$.

Taking for the path of integration a line $J_2'$ similar to $J_1'$ but crossing the quadrant containing $-i\sqrt{\sigma}$ in the reverse direction, we obtain the solution

$$z_2(x) = \int_{J_2'} t^x e^{\frac{\alpha t}{\sigma}} e^{-\frac{t^2}{2\sigma}}\, dt;$$

here we will let $\arg t$ have values between $\frac{1}{2}\arg\sigma$ and $\frac{1}{2}\arg\sigma - \pi$. If we set

$$t = -\sqrt{\frac{2\sigma}{\tau}} = e^{-\pi i}\sqrt{\frac{2\sigma}{\tau}},$$

then

$$z_2(x) = -\frac{1}{2}e^{-\pi i x}(2\sigma)^{\frac{x+1}{2}}\int_K \tau^{-\frac{x}{2}-\frac{3}{2}} e^{-\frac{\sqrt{2}\,\varkappa}{\sqrt{\tau}}} e^{-\frac{1}{\tau}}\, d\tau$$

$$= -\frac{1}{2}e^{-\pi i x}(2\sigma)^{\frac{x+1}{2}}\left[\Gamma\left(\frac{x}{2}+\frac{1}{2}\right) - \sqrt{2}\,\varkappa\,\Gamma\left(\frac{x}{2}+1\right) + \frac{2\varkappa^2}{2!}\Gamma\left(\frac{x}{2}+\frac{3}{2}\right) - \cdots\right]$$

$$= \frac{1}{2}e^{-\pi i x}(2\sigma)^{\frac{x+1}{2}}\left\{-\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)\left[1+\frac{\varkappa^2}{2!}(x+1) + \frac{\varkappa^4}{4!}(x+1)(x+3)+\cdots\right] + \sqrt{2}\,\Gamma\left(\frac{x}{2}+1\right)\left[\varkappa+\frac{\varkappa^3}{3!}(x+2) + \frac{\varkappa^5}{5!}(x+2)(x+4)+\cdots\right]\right\}.$$

Instead of the lines $J_1'$ and $J_2'$, we can use loops $L_1$ and $L_2$ which start and end at $t = \infty$ in the quadrants containing $-\sqrt{\sigma}$ and $\sqrt{\sigma}$ respectively and pass around $t = 0$ in the negative direction; these yield the solutions $z_1(x) + z_2(x)$ and $z_1(x) + e^{2\pi i x} z_2(x)$.

If we integrate along a straight line from 0 to $\infty$ in the sector containing $\sqrt{\sigma}$, we obtain the solution

$$z_3(x) = \int_0^\infty t^x e^{\frac{\alpha t}{\sigma}} e^{-\frac{t^2}{2\sigma}} dt,$$

valid for $R(x) > -1$; we will let $\arg t$ have a value between $\frac{1}{2}\arg\sigma - \pi/4$ and $\frac{1}{2}\arg\sigma + \pi/4$. Setting $t = \sqrt{2\sigma\tau}$, we have

$$\begin{aligned}
z_3(x) &= \frac{1}{2}(2\sigma)^{\frac{x+1}{2}} \int_0^\infty \tau^{\frac{x}{2}-\frac{1}{2}} e^{\sqrt{2}\varkappa\sqrt{\tau}} e^{-\tau} d\tau \\
&= \frac{1}{2}(2\sigma)^{\frac{x+1}{2}} \left[\Gamma\left(\frac{x}{2}+\frac{1}{2}\right) + \sqrt{2}\,\varkappa\,\Gamma\left(\frac{x}{2}+1\right)\right. \\
&\qquad\qquad \left. + \frac{2\varkappa^2}{2!}\Gamma\left(\frac{x}{2}+\frac{3}{2}\right) + \cdots\right] \\
&= \frac{1}{2}(2\sigma)^{\frac{x+1}{2}} \left\{\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)\left[1 + \frac{\varkappa^2}{2!}(x+1)\right.\right. \\
&\qquad\qquad \left. + \frac{\varkappa^4}{4!}(x+1)(x+3) + \cdots\right] \\
&\qquad\qquad + \sqrt{2}\,\Gamma\left(\frac{x}{2}+1\right)\left[\varkappa + \frac{\varkappa^3}{3!}(x+2)\right. \\
&\qquad\qquad \left.\left. + \frac{\varkappa^5}{5!}(x+2)(x+4) + \cdots\right]\right\}.
\end{aligned}$$

Integrating similarly along a line from 0 to $\infty$ in the sector containing $-\sqrt{\sigma}$, we have the solution

$$z_4(x) = \int_0^\infty t^x e^{\frac{\alpha t}{\sigma}} e^{-\frac{t^2}{2\sigma}} dt,$$

in which we will let $\arg t$ have a value between $\frac{1}{2}\arg\sigma + 3\pi/4$ and $\frac{1}{2}\arg\sigma + 5\pi/4$. If we set

$$t = -\sqrt{2\sigma\tau} = e^{\pi i}\sqrt{2\sigma\tau},$$

then

$$z_4(x) = -\frac{1}{2}e^{\pi i x}(2\sigma)^{\frac{x+1}{2}}\int_0^\infty \tau^{\frac{x}{2}-\frac{1}{2}}e^{-\sqrt{2}\,\varkappa\sqrt{\tau}}\,e^{-\tau}\,d\tau$$

$$= -\frac{1}{2}e^{\pi i x}(2\sigma)^{\frac{x+1}{2}}\Big[\Gamma\Big(\frac{x}{2}+\frac{1}{2}\Big)-\sqrt{2}\,\varkappa\,\Gamma\Big(\frac{x}{2}+1\Big)$$
$$+\frac{2\varkappa^2}{2!}\Gamma\Big(\frac{x}{2}+\frac{3}{2}\Big)+\cdots\Big]$$

$$= \frac{1}{2}e^{\pi i x}(2\sigma)^{\frac{x+1}{2}}\Big\{-\Gamma\Big(\frac{x}{2}+\frac{1}{2}\Big)\Big[1+\frac{\varkappa^2}{2!}(x+1)$$
$$+\frac{\varkappa^4}{4!}(x+1)(x+3)+\cdots\Big]$$
$$+\sqrt{2}\,\Gamma\Big(\frac{x}{2}+1\Big)\Big[\varkappa+\frac{\varkappa^3}{3!}(x+2)$$
$$+\frac{\varkappa^5}{5!}(x+2)(x+4)+\cdots\Big]\Big\}.$$

By dividing these solutions of eq. (246) by $\Gamma(x+1)$ or $\overline{\Gamma}(x+1)$, we obtain solutions of eq. (235). If we write

$$f_1(x) = 1+\frac{\varkappa^2}{2!}(x+1)+\frac{\varkappa^4}{4!}(x+1)(x+3)+\cdots,$$

$$f_2(x) = \varkappa+\frac{\varkappa^3}{3!}(x+2)+\frac{\varkappa^5}{5!}(x+2)(x+4)+\cdots,$$

these solutions may be taken as

$$G(x) = \frac{z_1(x)}{\overline{\Gamma}(x+1)} =$$
$$\frac{(2\sigma)^{\frac{x+1}{2}}}{2\overline{\Gamma}(x+1)}\Big[\overline{\Gamma}\Big(\frac{x}{2}+\frac{1}{2}\Big)f_1(x)+\sqrt{2}\,\overline{\Gamma}\Big(\frac{x}{2}+1\Big)f_2(x)\Big],$$

$$G'(x) = \frac{z_2(x)}{\overline{\Gamma}(x+1)} =$$
$$\frac{e^{-\pi i x}(2\sigma)^{\frac{x+1}{2}}}{2\overline{\Gamma}(x+1)}\Big[-\overline{\Gamma}\Big(\frac{x}{2}+\frac{1}{2}\Big)f_1(x)+\sqrt{2}\,\overline{\Gamma}\Big(\frac{x}{2}+1\Big)f_2(x)\Big],$$

$$H(x) = \frac{z_3(x)}{\Gamma(x+1)} =$$

$$\frac{(2\sigma)^{\frac{x+1}{2}}}{2\Gamma(x+1)}\left[\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)f_1(x) + \sqrt{2}\,\Gamma\left(\frac{x}{2}+1\right)f_2(x)\right].$$

$$H'(x) = \frac{z_4(x)}{\Gamma(x+1)} =$$

$$\frac{e^{\pi i x}(2\sigma)^{\frac{x+1}{2}}}{2\Gamma(x+1)}\left[-\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)f_1(x) + \sqrt{2}\,\Gamma\left(\frac{x}{2}+1\right)f_2(x)\right].$$

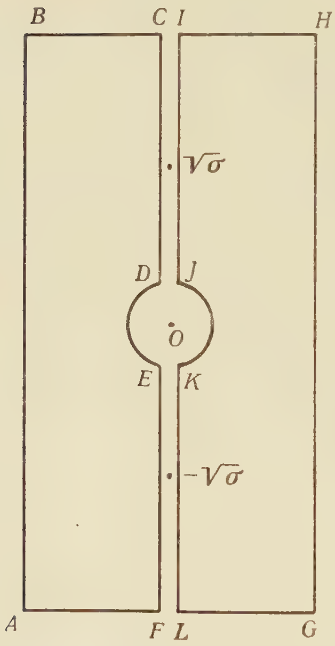


Fig. 29.

$G(x)$ and $G'(x)$ have poles at $x = 0, 1, 2, 3, \cdots$, while $H(x)$ and $H'(x)$ are entire functions.

To obtain the relations between these four solutions, let us integrate over the contour $ABCDEFA$ of Fig. 29, starting with $\arg t$ near $\frac{1}{2}\arg\sigma + \pi$. If $R(x) > 0$ we can let the radius of the arc $DE$ approach 0; we can also let the points $B$, $C$, $A$, and $F$ move off indefinitely far; then we have in the limit

$$z_1(x) - z_3(x) + z_4(x) = 0.$$

Integrating similarly over the contour $GHIJKLG$, starting with $\arg t$ near $\frac{1}{2}\arg\sigma - \pi$, we have

$$-z_2(x) - z_3(x) + e^{-2\pi i x} z_4(x) = 0.$$

These relations are identities between analytic functions, so we can drop the restriction $R(x) > 0$.

From these equations follow the relations

$$G(x) = \frac{H(x) - H'(x)}{1 - e^{2\pi i x}}, \qquad G'(x) = \frac{-H(x) + e^{-2\pi i x} H'(x)}{1 - e^{2\pi i x}};$$

$$H(x) = G(x) + e^{2\pi i x} G'(x), \qquad H'(x) = e^{2\pi i x}[G(x) + G'(x)].$$

We have now considered all the irregular cases of the hypergeometric difference equation, though it will be seen that much remains to be done in the way of coördinating the results of the present section into a systematic theory of the case where both roots of the characteristic equation are zero.

The student who desires to go further is now in a position to read the original memoirs on the theory of linear difference equations. The general theory of systems of $n$ linear homogeneous difference equations of the first order, based on the matrix method used in Chapters III and IV, is contained in a paper by Birkhoff (General Theory of Linear Difference Equations, Trans. Amer. Math. Soc. 12 (1911), pp. 243–284). The corresponding theory for a single equation of the $n$th order was worked out by Carmichael (On the Solutions of Linear Homogeneous Difference Equations, Amer. Journ. Math. 38 (1916), pp. 185–220). A treatment of non-homogeneous equations by Birkhoff's methods has been given by Williams (The Solutions of Non-homogeneous Linear Difference Equations and their Asymptotic Form, Trans. Amer. Math. Soc. 14 (1913), pp. 209–240). On the other hand, fundamental use of the Laplace transformation is made by Nörlund, who bases his study of difference equations primarily on the properties of the differential equations into which they are transformed (Sur les équations linéaires aux différences finies à coefficients rationnels, Acta Math. 40 (1915), pp. 191–249). In a recent book (Vorlesungen über Differenzenrechnung, Berlin, 1924) Nörlund has summarized the results of numerous investigators who have dealt with problems more or less closely connected with linear difference equations, with full references to the sources.

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